Partially identified heteroskedastic SVARs: an application to the market for crude oil

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Abstract

This paper studies the identification of Structural Vector Autoregressions (SVARs) exploiting a break in the variances of the structural shocks. Point-identification for this class of models relies on an eigen-decomposition involving the covariance matrices of reduced-form errors and requires that all the eigenvalues are distinct. This point-identification, however, fails in the presence of multiplicity of eigenvalues. This occurs in an empirically relevant scenario where, for instance, only a subset of structural shocks had the break in their variances, or where a group of variables shows a variance shift of the same amount. Together with zero or sign restrictions on the structural parameters and impulse responses, we derive the identified sets for impulse responses and show how to compute them. We perform inference on the impulse response functions, building on the robust Bayesian approach developed for set identified SVARs. To illustrate our proposal, we present an empirical example based on the literature on the global crude oil market where the identification is expected to fail due to multiplicity of eigenvalues.

Keywords: SVARs with heteroskedastic shocks, identification, robust Bayesian approach, credible region.
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I Introduction

In recent years many researchers have stressed the importance of external information to identify the parameters of the econometric models. In this respect, information coming from the data or from external proxies can be considered along with traditional approaches based on restricting the parameter space through constraints derived from economic theory, that are often hard to defend.

In the last two decades the literature has benefited from the idea that the heteroskedasticity present in the data can add important information to the identification of simultaneous equations systems (Rigobon, 2003). When volatility clusters observed in the data can be attributed to shifts in the variance of the structural innovations – while leaving the parameters of the conditional mean constant – there are important gains in terms of identification of the structural parameters. Lanne and Lütkepohl (2008) extended this intuition to the class of structural vector autoregressions (SVARs) and opened the door to many contributions that has made this approach a standard tool in macroeconometrics (see, e.g., Kilian and Lütkepohl, 2017, Chapter 14, or the recent empirical work by Brunnermeier, Palia, Sastry, and Sims, 2021).

The benefits of heteroskedasticity for the identification of structural shocks in SVARs are well known, but two caveats apply. First, the identification through heteroskedasticity is a pure statistical identification scheme. In other words, shocks have a structural interpretation only if they yield impulse responses with a credible economic interpretation. Second, the size of the shifts in the variance of different shocks must be heterogeneous enough for the validity of identification. Lütkepohl et al. (2020) developed a formal test for identification through heteroskedasticity. The test checks whether all the eigenvalues of an eigen-decomposition problem are statistically distinct. When this is the case, information provided by the data is sufficient for the identification. It is thus possible to use this unrestricted representation of the model as a starting point for comparison with alternative, restricted, specifications characterized by competing identifying restrictions. In other words, heteroskedasticity makes overidentifying testable restrictions readily available (see Lanne and Lütkepohl, 2008; Lütkepohl and Netšunajev, 2017). On the contrary, when some of the eigenvalues are not distinct, the model cannot be identified through heteroskedasticity. In this case, the literature does not offer any solution and other identification schemes – provided that they are credible – must be considered because otherwise the model would not be identified.

This paper provides a new strategy that allows researchers to rely on structural vector autoregressions with heteroskedasticity (HSVARs) even in the absence of information to point identify the shocks of interest. Our approach starts where the test by Lütkepohl et al. (2020) suggests to stop, i.e. when some of the eigenvalues, that can be interpreted as the shifts in the variances of the structural shocks, are not statistically distinguishable from each other. The idea is to combine the presence of heteroskedasticity with some zero and sign restrictions on parameters or on functions of them. This strategy allows to deal with heteroskedastic SVARs that are not point identified, as common in the literature, but only set identified.

The first contribution of the paper, thus, is to extend the literature on set identification, largely used in SVAR models, to HSVARs. To the best of our knowledge, this is completely new in the literature and offers applied economists and econometricians a new tool for conducting their empirical analysis when clusters of volatility offer a useful but not sufficient source of information for the identification of structural shocks.

The second contribution, involves analytical results on identification, that are developed starting from a geometrical interpretation of the identification issue in HSVARs, when compared to standard SVARs. This geometrical interpretation is facilitated by the strategy of thinking about the structural parameters of the model as a unit rotation of those of the reduced form. This way of mapping the reduced- and structural-form parameters allows to graphically see how set identification in SVARs becomes point identification in HSVARs, and to what extent proportional shifts in the variances reduce the amount of information contained in the data that can be exploited for identification. Specifically, we show that apart from normalization constraints, a combination of heteroskedasticity and zero restrictions can point identify the HSVAR, even in the case of lack of heterogeneous variance shifts. The number of zero restrictions is much smaller than the one generally used for point identification in traditional SVARs.

Strictly connected to the previous two contributions, we extend the topological analysis of the identified set offered in Giacomini and Kitagawa (2021) for SVARs to the case of HSVARs, and derive analytical results for point identification and for set identification with convex identified sets.

Finally, a further contribution to the literature is about the estimation and inference on the identified set. In this respect, we first derive a Bayesian estimator based on a Gibbs sampler and independent priors for the reduced-form parameters of the model, and then adapt the robust Bayes approach of Giacomini and Kitagawa (2021) to our setup. This approach, in fact, perfectly fits the peculiarities of HSVARs featuring failures of the identifying assumptions on heterogeneous variance shifts in the structural shocks. We provide a useful algorithm to implement our strategy for estimation and inference of the identified set.

An empirical example about the identification of structural shocks driving the real price of crude oil illustrates our methodology, from the failure of distinct eigenvalues observed in the data, to the implementation of the algorithm combining heteroskedasticity and parameter restrictions to point or set identify the objects of interest, in general impulse responses. In this example we show how to set or point identify structural shocks when the standard HSVAR approach fails because of multiplicity in eigenvalues and highlight that thanks to our methodology this can be accomplished with less restrictions that in standard SVAR models.

The rest of the paper is organized as follows. The next section briefly surveys the literature; Section II introduces the econometric framework and provides some preliminary results on the identification of HSVARs. Section III and Section IV are dedicated to the theory of identification in HSVARs. Section V focuses on the inferential analysis of identified sets through a Robust Bayesian approach. Section VI presents the empirical example and Section VII concludes. An appendix with further results and proofs completes the paper.

I.1 Related literature

This paper is strongly related to the identification through heteroskedasticity literature both for simultaneous equations systems (Rigobon, 2003; Klein and Vella, 2010; Lewbel, 2012) and SVARs (Lanne and Lütkepohl, 2008; Bacchiocchi, 2017 and all references in Kilian and Lütkepohl, 2017, Chapter 14, and in Lütkepohl and Netšunajev, 2017). However, to the best of our knowledge, the idea that heteroskedasticity can be helpful in identifying econometric models has been firstly proposed by Sentana and Fiorentini (2001) in a context of factor models, nesting SVAR models as well. Our contribution builds on the results of the statistical test for heterogeneous variance shifts by Lütkepohl et al. (2020), in the sense that our approach can be implemented when, as an example, the test of Lütkepohl et al. (2020), or any alternative one, suggests a failure of the identifying information from heteroskedasticity to point identify the structural shocks. Different approaches exploiting heteroskedasticity for the identification of structural shocks are the recent contributions by Lewis (2021), that does not require volatility clusters, but simply needs for time-varying volatility of unknown form, as well as Lütkepohl and Schlaak (2021), who analyse identification through heteroskedasticity in the context of proxy VAR models (see also Sims, 2020, for a recent contribution on heteroskedastic SVARs with misspecified regimes).

While in our contribution the main assumption is that only the variances of the shocks are subject to breaks, other authors found evidence of structural shifts among the structural parameters of the model, too (see, among others Sims and Zha, 2006; Inoue and Rossi, 2011; Boivin and Giannoni, 2006). This literature, that does not focus solely on SVAR models, allows for impulse responses to be different in the different regimes, while they are the same when only the variances do change. However, only few papers constructively use the presence of regime shifts to solve the identification issue (Magnusson and Mavroeidis, 2014; Bacchiocchi and Fanelli, 2015; Bacchiocchi and Kitagawa, 2020b).

This paper contributes also to the literature on point and set identified SVARs. For point identification, we exploit the general criteria in Rubio-Ramírez, Waggoner, and Zha (2010), with subsequent modifications proposed by Bacchiocchi and Kitagawa (2021) for global identification on SVARs. As for set identification, we use sign restrictions à la Uhlig (2005) to set identify the structural impulse response functions of interest.

Strictly connected to this last point is the literature on how to do inference on set identified models. As already said in the introduction, our approach builds on Giacomini and Kitagawa (2021), but other approaches have been proposed in the literature to pursue this purpose. Gafarov, Meier, and Montiel-Olea (2018) and Granziera, Moon, and Schorfheide (2018) provide results based on a frequentist setting, while, among others, Baumeister and Hamilton (2015) and Arias, Rubio-Ramírez, and Waggoner (2018) adopt Bayesian inference. Baumeister and Hamilton (2015), caution on the use of standard Bayesian approaches based on setting priors also on the rotation matrices, that, although non-informative, are unrevisable and as such may not vanish on the posterior even asymptotically. The robust Bayesian inference adopted in our paper overcomes this drawback by specifying not just one prior, but a class of priors for each single prior for the reduced-form parameters, as proposed by Giacomini and Kitagawa (2021).

II SVARs and HSVARs: definitions and preliminary results

II.1 Econometric framework

Consider the following Structural Vector Autoregressive (SVAR) model

$$A_0 y_t = a + \sum_{i=1}^{l} A_i y_{t-i} + \varepsilon_t \tag{1}$$

where y_t is a n-dimensional vector of observable variables, ε_t is a vector of mutually orthogonal white noise processes, normally distributed with mean zero and time-varying covariance matrix. Specifically, let the covariance matrix of the structural shocks ε_t be as follows

$$E(\varepsilon_t \varepsilon_t') = \begin{cases} I_n & \text{if} \quad 1 \leq t \leq T_B \\ \Lambda & \text{if} \quad T_B < t \leq T \end{cases} \tag{2}$$

where $1 < T_B < T$ is the break date, I_n is the $(n \times n)$ identity matrix, and Λ is a $(n \times n)$ diagonal matrix made of strictly positive numbers.[1] The $n \times 1$ vector a contains the intercepts and the $n \times n$ matrices A_i, with $i = 0, \ldots, l$, collect the structural parameters. The structural parameters can be indicated as $\theta = (A_0, A_+, \Lambda) \in \Theta \subset \mathbb{R}^{(n+m)n+n}$, with $m = nl + 1$, and where the $n \times m$ matrix $A_+ = (a, A_1, \ldots, A_l)$. We denote the open dense set of all structural parameters by $\mathbb{P}^S \subset \mathbb{R}^{(n+m)n+n}$. The model in Eq.s (1)-(2) is a standard SVAR model with structural shocks characterized by different volatility regimes. As shown in Eq. (2), structural innovations have unit variance before the break, and variance equal to the diagonal elements of Λ, denoted as λ_i after the break. Hereafter, this model is referred to as heteroskedastic SVAR (HSVAR).

The reduced-form VAR model can be written as

$$y_t = b + \sum_{i=1}^{l} B_i y_{t-i} + u_t \tag{3}$$

where $b = A_0^{-1} a$, $B_i = A_0^{-1} A_i$. Furthermore, for both the regimes, the vector of error terms is defined as $u_t = A_0^{-1} \varepsilon_t$, with

$$E(u_t u_t') = \begin{cases} \Omega_1 = A_0^{-1} A_0^{-1\prime} & \text{if} \quad 1 \leq t \leq T_B \\ \Omega_2 = A_0^{-1} \Lambda A_0^{-1\prime} & \text{if} \quad T_B < t \leq T \end{cases} \tag{4}$$

The VAR model, thus, presents different covariance matrices of the error terms Ω_1 and Ω_2, and thus heteroskedasticity, as in, among others, Rigobon (2003), Lanne and Lütkepohl (2008), and Bacchiocchi and Fanelli (2015). The reduced-form parameters are denoted by $\phi = (B, \Omega_1, \Omega_2) \in \Phi \subset \mathbb{R}^{n+n^2 l} \times \Omega_n \times \Omega_n$, where the $n \times m$ matrix $B = (b, B_1, \ldots, B_l)$ and Ω_n is the space of positive-semidefinite matrices of dimension $n \times n$. The set of all reduced-form parameters is denoted by $\mathbb{P}^R \subset \mathbb{R}^{nm+n(n+1)}$. The reduced form will be denoted HVAR.

Conditional on the restrictions of the domain Φ such that all the roots of the characteristic polynomial lie outside the unit circle, there exists an equivalent VMA(∞) representation for the HVAR in Eq. (3), assuming the form

$$\begin{aligned} y_t &= c + \sum_{j=0}^{\infty} C_j u_{t-j} \\ &= c + \sum_{j=0}^{\infty} C_j A_0^{-1} \varepsilon_{t-j} \end{aligned} \tag{5}$$

where C_j is the j-th coefficient matrix of $\left(I_n - \sum_{i=1}^{l} B_i L^i \right)^{-1}$.

[1] We consider the initial conditions for the first regime, $y_0, \ldots, y_{1-l}$, as given, while for the second regime they are fixed as the last l observations of the former, in order to guarantee the contiguity of the regimes on the whole sample.

Remark 1. The HVAR in Eq.s (3)-(4) is characterized by the same parameters for the conditional mean over the two regimes, therefore breaks are confined to second moments parameters. As a consequence, the absence of unit roots is a characteristic of the model in the whole sample, and not within each regime. Furthermore, as shown in Eq. (5), the VMA representation is unique and not regime-specific. It follows that, if shocks have the same magnitude, impulse response functions in the two regimes are equivalent. If instead one considers a one standard deviation structural shock, impulse responses will be of different magnitude, but have exactly the same shape in different regimes.

Based on the VMA representation, the impulse response IR^h is

$$IR^h = C_h A_0^{-1} \tag{6}$$

whose (i,j)-element represents the response of the i-th variable of y_{t+h} to a *unit* shock on the j-th element of ε_t, independently of the regime considered. The long-run impulse response matrix is defined as

$$IR^\infty = \lim_{h\to\infty} IR^h = \left(I_n - \sum_{j=1}^{l} B_j\right) A_0^{-1} \tag{7}$$

and the long-run cumulative impulse response matrix becomes

$$CIR^\infty = \sum_{h=0}^{\infty} IR^h = \left(\sum_{h=0}^{\infty} C_h\right) A_0^{-1}. \tag{8}$$

II.2 Preliminary results on the identification of HSVARs

As is well known in the SVAR literature, without any restriction it is impossible to uniquely pin down the structural parameters based on the reduced form of the model. If, instead, we suppose the parameters of the conditional mean in the HSVAR in Eq. (1) to remain stable across the two regimes, then Rigobon (2003), for a bivariate case, and Lanne and Lütkepohl (2008), for the general case, proved there is some gain in terms of identification. This section deals with the identification issue in HSVARs. We first provide some general theoretical results and then focus on the bivariate case, for which the theoretical results can also be clearly represented in graph. We start by providing results on bivariate SVAR models that will be extremely useful when moving to bivariate HSVARs. The generalization to larger HSVARs is discussed in Section IV, where we introduce zero and sign restricts, we show results on the point identification of structural shocks and provide insights on the topology of the identified set when point identification fails.

Some of the theoretical results we present in this section are not completely new in the literature, although they have been derived independently from other authors. However, we have decided to report them as prerequisites for a better understanding of the main results provided in the paper. Detailed references will be reported accordingly.

Consider an n-variable SVAR model with two regimes in structural shock variances, but maintaining the homogeneity of the structural coefficients. We normalize the covariance matrix of the structural shocks to $n \times n$ identity matrix in the first regime. Following the parametrization and notations we have been using so far, we analyze identification of the $n \times n$ matrix C that represents the inverse of structural coefficient matrix A_0, i.e. $C \equiv A_0^{-1}$, and Λ, $n \times n$ diagonal matrix with strictly positive elements. Given the reduced-form covariance matrix at regime 1 and 2, denoted by Ω_1 and Ω_2, respectively, C and Λ solve

$$\Omega_1 = CC', \tag{9}$$
$$\Omega_2 = C\Lambda C' \tag{10}$$

The next theorem characterizes the set of (C, Λ) solving this equation system. To state it, we define $\mathcal{P}(n)$ as the set of $n \times n$ permutation matrices, such that pre-multiplying $P \in \mathcal{P}(n)$ to any matrix M performs a row-permutation of M, and post-multiplying it performs a column-permutation. Moreover, in the case of a diagonal matrix D of size n, $P'DP$ performs a permutation of the diagonal elements of D. Let $\mathcal{D}(n)$ be the set of $n \times n$ diagonal matrices whose diagonal entries are either $+1$ or -1. That is, if the i-th diagonal entry of $S \in \mathcal{D}(n)$ is -1, pre-multiplying (post-multiplying) S to any matrix M flips the sign of the i-th row (resp. column) vector of M.

Theorem 1 (Sign normalization and column-permutation)**.** *Assume Ω_1 and Ω_2 are non-singular. Suppose (C^*, Λ^*) is a solution of the equation system (9)-(10). Then, the set of solutions solving (9)-(10) is represented as*

$$\left\{(C, \Lambda) = (C^*SP, P\Lambda^*P') : P \in \mathcal{P}(n), S \in \mathcal{D}(n)\right\}. \tag{11}$$

Proof. See the appendix. □

This theorem clarifies the fundamental indeterminacy of the solutions in the equation system (9)-(10). Specifically, the representation of the solutions in Eq. (11) shows that (C, Λ) remains observationally equivalent with respect to any permutation and change of signs of the column vectors in C as far as the same permutation is applied to the diagonal elements of Λ. The observational equivalence with respect to $S \in \mathcal{D}(n)$ corresponds to the indeterminacy of the signs of structural shocks common in any SVAR modelling (see Lanne, Lütkepohl, and Maciejowska, 2010, for an equivalent result on HSVARs). We often control such sign indeterminacy by imposing the sign normalization restrictions that pin down S, e.g., restricting the diagonal elements of $A_0 = C^{-1}$ to be non-negative. The observational equivalence with respect to the permutations corresponds to the indeterminacy of the structural parameters with respect to the reordering of the structural equations. Rigobon (2003) noted this indeterminacy of the ordering of the structural equations in bivariate HSVAR models and argued that sign restrictions placed on the off-diagonal elements of $A_0 = C^{-1}$ resolve such indeterminacy.

Theorem 1 implies that with sign normalization restrictions imposed, point-identification of (C, Λ) requires an assumption that pins down the ordering of the equations (i.e., permutation matrix P). One way to constrain the ordering of the equations is to exploit available knowledge on the ratios of the structural shock variances of regime 1 to regime 2. In particular, assuming a complete ordering of the structural shocks according to their variance ratios can fix the order of the structural equations based on the diagonal entries of the true Λ. Hence, if a solution of Λ is such that all of its diagonal elements are distinct, a complete ordering of such elements reduces the set of solutions in Eq. (11) to a singleton. The following theorem hence follows as a corollary of Theorem 1.

Theorem 2 (Point identification)**.** *In addition to the assumptions of Theorem 1, assume that a solution of Λ has the diagonal terms all distinct. Then, with sign normalization restrictions and complete ordering of the structural shocks according to the variance ratios imposed, (C, Λ) is point-identified.*

Proof. The theorem, apart from the potential indeterminacy due to the column permutation, corresponds to Proposition 1 in Lanne, Lütkepohl, and Maciejowska (2010), and can be proved in exactly the same way. However, according to Theorem 1 here before, fixing the ordering of the shocks is also necessary in order to have point identified (C, Λ). □

If a solution of Λ has some of the diagonal elements identical, then invariance of Λ with respect to a permutation that permutes only these elements fails to uniquely pin down C within the set of solutions in Eq. (11). Partial identification of C matrix in this case is to be considered below.

The identification result of Theorem 2 is not constructive and it does not provide an explicit analytical expression of (C, Λ) as a function of (Ω_1, Ω_2). A more constructive identification result for (C, Λ) can be obtained by representing the equation systems (9) and (10) as a certain eigen-decomposition problem.

Let $\Omega_{1,tr}$ be a lower triangular Cholesky decomposition of Ω_1. Following Proposition A.1 of Uhlig (2005), the set of non-singular matrices solving Eq. (9) can be expressed as $C = \Omega_{1,tr}Q$, $Q \in \mathcal{O}(n)$, where $\mathcal{O}(n)$ is the set of $n \times n$ orthogonal matrices. Plugging this representation of C into Eq. (10), leads to

$$\begin{aligned} &C = \Omega_{1,tr}Q \\ &\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime} = Q\Lambda Q'. \end{aligned} \tag{12}$$

Symmetry of $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$ and orthogonality of Q implies that solving Eq. (12) is precisely the eigen-decomposition problem. Identification of (C, Λ) can be therefore cast as uniqueness of the eigen-decomposition of $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$ into the diagonal matrix of eigenvalues and the corresponding eigenvectors collected in Q. This perspective yields the following succinct analytical characterization of the solutions of the equation system.

Theorem 3 (Identification and eigen-decomposition)**.** *The set of solutions solving system (9)-(10) can be represented by $(\Omega_{1,tr}Q, \Lambda)$, where Λ is a diagonal matrix of eigenvalues of $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$ and Q is an orthogonal matrix of the corresponding eigenvectors. Hence, if the eigenvalues of $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$ are all distinct, (C, Λ) is identified up to permutations and sign changes of the structural equations.*

Proof. The result immediately follows from Theorem A9.9, and the related proof, in Muirhead (1982). □

The claim of this theorem simplifies computation of an estimator of (C, Λ); the maximum likelihood estimator for (C, Λ) can be computed by performing an eigen-decomposition on the maximum likelihood estimator of $\Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$ subject to the sign normalization. If a complete ordering assumption on Λ is available (e.g., the diagonal elements of Λ is decreasing), we can obtain a point-estimator for (C, Λ) by ordering the eigenvalues accordingly through the decomposition. If the ordering assumption is not available, then permutations of the diagonal elements in Λ and the corresponding eigenvectors in Q span the identified set of (C, Λ). The idea of treating identification and estimation of HSVARs as an eigen-decomposition issue has been also pursued by Lütkepohl, Meitz, Netšunajev, and Saikkonen (2020), that developed their test for identification via heteroskedasticity as a test on equivalent eigenvalues.

An alternative way to see and address the identification problem of (C, Λ) is to look at system (9)-(10) in a slightly different way. In fact, given that Λ is made of positive elements, it is possible to rewrite Eq. (10) as $\Omega_2 = C\Lambda^{1/2}\Lambda^{1/2}C'$. The quantity $C\Lambda^{1/2}$ could not be unique because of the presence of an orthogonal matrix Q_2 such that $\Omega_2 = C\Lambda^{1/2}Q_2Q_2'\Lambda^{1/2}C'$. Using the result in Proposition A.1 of Uhlig (2005) for the decomposition of Ω_1 and Ω_2 yields the following system

$$\begin{aligned} C &= \Omega_{1,tr}Q_1 \\ \Omega_{2,tr} &= C\Lambda^{1/2}Q_2 \end{aligned}$$

and plugging the definition of C into the second equation we obtain

$$\Omega_{1,tr}^{-1}\Omega_{2,tr} = Q_1\Lambda^{1/2}Q_2. \tag{13}$$

The next theorem discusses the identification issue of the structural parameters (C, Λ) in terms of the uniqueness of Q_1 and Q_2.

Theorem 4 (Identification and Single Value Decomposition)**.** *The set of solutions of system (9)-(10) can be represented by $(\Omega_{1,tr}Q_1, \Lambda)$, where Λ is a diagonal matrix made of positive elements and Q_1 is an orthogonal matrix solving the Single Value Decomposition of $\Omega_{1,tr}^{-1}\Omega_{2,tr} = Q_1\Lambda^{1/2}Q_2$. If the entries in Λ are all distinct, then Q_1 and Q_2 are unique apart from simultaneous sign changes and permutation of their corresponding columns. Hence, (C, Λ) is identified up to permutations and sign changes of the structural equations.*

Proof. See the Appendix A. □

III Geometry of identification in bivariate SVARs and HSVARs

This section is dedicated to the identification issue in bivariate SVAR models. We first introduce the notion of set identification in standard SVARs as in Giacomini and Kitagawa (2021), and derive point identification as a particular case. Second, we move to the core of the paper and extend the set and point identification notions to SVARs characterized by structural breaks, that, as shown in Bacchiocchi and Fanelli (2015), is more general than the separate analysis of each single regime.

Consider the following bivariate model, where, for simplicity, the dynamics is omitted, as not directly involved in the identification issue:

$$\begin{pmatrix} 1 & -\beta \\ -\alpha & 1 \end{pmatrix}\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} \tag{14}$$

or, more compactly,

$$AY_t = \epsilon_t. \tag{15}$$

In order to ease the explanation, we introduce a theoretical foundation to the model and interpret the first equation as a demand equation while the latter as a supply equation. The vector $Y_t = (p_t\,,\,q_t)'$ collects the two observable variables and $\epsilon_t = (\varepsilon_t\,,\,\eta_t)'$ the two structural shocks. Furthermore, let the structural shocks be characterized by null expected values and by the following covariance matrix:

$$\Sigma \equiv \mathrm{Cov}\begin{pmatrix} \varepsilon_t \\ \eta_t \end{pmatrix} = \begin{pmatrix} \sigma_\varepsilon^2 & 0 \\ 0 & \sigma_\eta^2 \end{pmatrix}.$$

The A matrix, containing the parameters of the simultaneous relationships among the observable variables, α and β, can also be rescaled by dividing for the standard deviations of the shocks and obtain

$$A_0 \equiv \Sigma^{-1/2} A = \begin{pmatrix} 1/\sigma_\varepsilon & -\beta/\sigma_\varepsilon \\ & \\ -\alpha/\sigma_\eta & 1/\sigma_\eta \end{pmatrix}.$$

Actually, from the observation of p_t and q_t, the amount of information is contained in the estimable covariance matrix

$$\Omega \equiv \mathrm{Cov}\begin{pmatrix} p_t \\ q_t \end{pmatrix} = \begin{pmatrix} \omega_p^2 & \omega_{pq} \\ \omega_{pq} & \omega_q^2 \end{pmatrix}$$

that is connected to the structural parameters through the non-linear system of equations

$$\begin{aligned} \Omega &= A^{-1}\Sigma A^{-1\prime} \\ &= A_0^{-1} A_0^{-1\prime} \end{aligned} \tag{16}$$

for which, when the solution with respect to the structural parameters is unique, the identification problem is clearly solved. As is well known, however, without imposing any restriction on the structural parameters, the solution cannot be unique as the three (estimable) empirical moments contained in Ω are not sufficient to consistently estimate the four structural parameters in A_0. In fact, the amount of information contained in Ω is the same as the one contained in the three elements of its lower triangular Cholesky factorization, that can be given by

$$\Omega_{1,tr} = \begin{pmatrix} \omega_p & 0 \\ \omega_{pq}/\omega_p & (\omega_q^2 - \omega_{pq}^2/\omega_p^2)^{1/2} \end{pmatrix}$$

whose inverse is given by

$$\Omega_{1,tr}^{-1} = \begin{pmatrix} \frac{1}{\omega_p} & 0 \\ -\frac{\omega_{pq}}{\omega_p^2}\left(\omega_q^2 - \frac{\omega_{pq}^2}{\omega_p^2}\right)^{-1/2} & \left(\omega_q^2 - \frac{\omega_{pq}^2}{\omega_p^2}\right)^{-1/2} \end{pmatrix} = (\omega_1\,,\,\omega_2) \tag{17}$$

where the two (2×1) vectors ω_1 and ω_2 are the two columns of $\Omega_{1,tr}^{-1}$.

According to Uhlig (2005), the identification issue can be seen in terms of the non uniqueness of an orthogonal matrix $Q \in \mathcal{O}(2)$, where $\mathcal{O}(2)$ is the set of (2×2) orthonormal matrices, such that $A_0 = Q'\Omega_{1,tr}^{-1}$ for which $\Omega = (A_0' A_0)^{-1}$. In other words, a unique Q guarantees a unique A_0 through a suitable rotation of $\Omega_{1,tr}^{-1}$, containing all the information coming from the reduced form parameters (or, better, the information coming from the data).

Denoting with $Q \equiv (q_1\,,\, q_2)$, with q_1 and q_2 being the columns of Q, the A_0 matrix can be given by

$$A_0 = Q'\Omega_{1,tr}^{-1} = \begin{pmatrix} q_1' \\ q_2' \end{pmatrix} (\omega_1\,,\, \omega_2)\,. \tag{18}$$

Moreover, we consider the following assumptions:

Assumption 1. (Sign normalization) Coherently with the sign normalization presented in Section IV.1, among all the possible rotations of $\Omega_{1,tr}^{-1}$ through the orthogonal matrix $Q \in \mathcal{O}(2)$, we consider only those guaranteeing the elements on the main diagonal of $A_0 = Q'\Omega_{1,tr}^{-1}$ to be positive. In different words, we consider only q_1 and q_2 such that $q_1'\omega_1 > 0$ and $q_2'\omega_2 > 0$.

Assumption 2. (Sign restriction on α and β) Coherently with the demand and supply curves in Eq. (14), we assume $\alpha \geq 0$ and $\beta \leq 0$.

The first assumption, that is standard in the SVAR literature, asserts that, as the product of $A_0^{-1}A_0^{-1\prime}$ is invariant to sign changes on the columns of A_0^{-1}, we select only $A_0 = Q'\Omega_{1,tr}^{-1}$ such that the elements on the main diagonal are strictly positive. Equivalently, we assume the shock to have a positive on impact effect on the corresponding observable variable. The second assumption, instead, refers to the economic interpretation of the two equations of the bivariate model in Eq. (14) as a demand and a supply equation, respectively.

III.1 Set identification in bivariate SVARs

Given the bivariate SVAR model discussed before, the following proposition provides the identification set for the two structural parameters α and β, according to the two potential cases of $\omega_{pq} \geq 0$ or $\omega_{pq} < 0$.

Theorem 5 (Set identification in bivariate SVARs)**.** *Given the bivariate model in Eq. (14), under Assumption 1, then:*

(Case I): if $\omega_{pq} \geq 0$, *then* $\alpha \in \left(-\infty\,;\, \frac{\omega_q^2}{\omega_{pq}}\right]$ *and* $\beta \in (-\infty\,;\,\infty)$*;*

(Case II): if $\omega_{pq} < 0$, *then* $\alpha \in (-\infty\,;\,\infty)$ *and* $\beta \in (-\infty\,;\,\infty)$*;*

under Assumption 1 and Assumption 2, then:

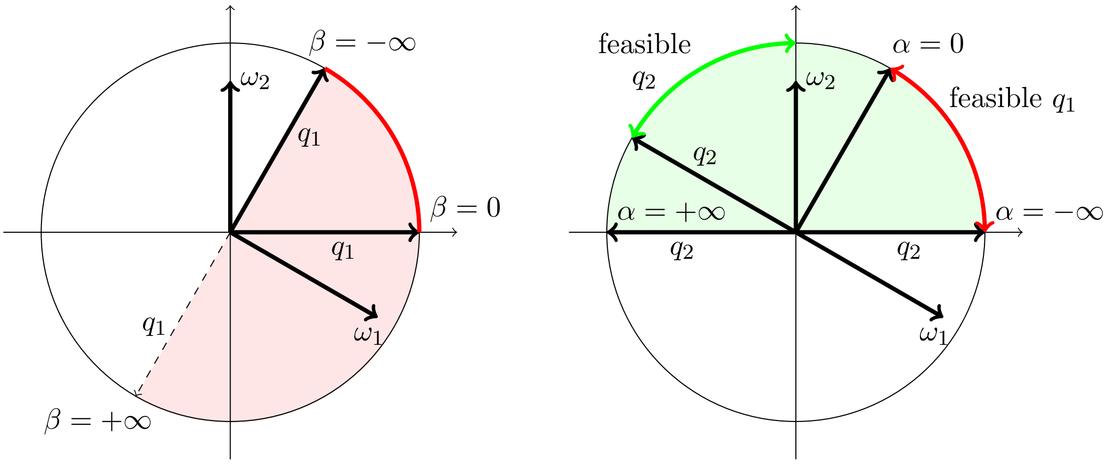
(Case I): if $\omega_{pq} \geq 0$, *then* $\alpha \in \left[\frac{\omega_{pq}}{\omega_p^2}\,;\, \frac{\omega_q^2}{\omega_{pq}}\right]$ *and* $\beta \in (-\infty\,;\,0]$*;*

(Case II): if $\omega_{pq} < 0$, *then* $\alpha \in [0\,;\,\infty)$ *and* $\beta \in \left[\frac{\omega_p^2}{\omega_{pq}}\,;\, \frac{\omega_{pq}}{\omega_q^2}\right]$.

Proof. See the Appendix B. □

As is well known, unless we impose at least one equality restriction on one of the structural parameters, the bivariate model cannot be point identified. More specifically, according to the recent contribution by Rubio-Ramírez, Waggoner, and Zha (2010), if either $\alpha = 0$ or $\beta = 0$ (homogeneous restrictions) are imposed, the model will be globally identified, otherwise, if any other non-homogeneous restriction is imposed, the model will be simply locally identified, see Bacchiocchi and Kitagawa (2020a). If no point restriction is imposed, the structural model remains unidentified, but focusing on the sign restrictions coherent with the theoretical interpretation of the model, together with the sign and magnitude of

Figure 1: Identification of α and β. Case I: $\omega_{pq} \geq 0$



Sign restriction: $\beta \leq 0$ Sign restrictions: $\beta \leq 0$ and $\alpha \geq 0$

Notes: Set identification of the parameter β (left panel) and joint set identification of α and β (right panel). The identified set, under the sign restriction consistent with a demand curve, i.e. $\beta < 0$, is represented by the red arc in both panels. In the right panel, the set identification of α under the further sign restriction consistent with a supply curve, i.e. $\alpha \geq 0$, is represented by the green arc. In both cases, the standard assumption of positive diagonal terms on A_0 is considered ($\sigma_\varepsilon > 0$ and $\sigma_\eta > 0$): in light red for the first equation and in light green for the second equation.

the elements in the reduced form covariance matrix among the observable variables, it is possible to obtain an identified set for α and β. This is what Theorem 5 reports.

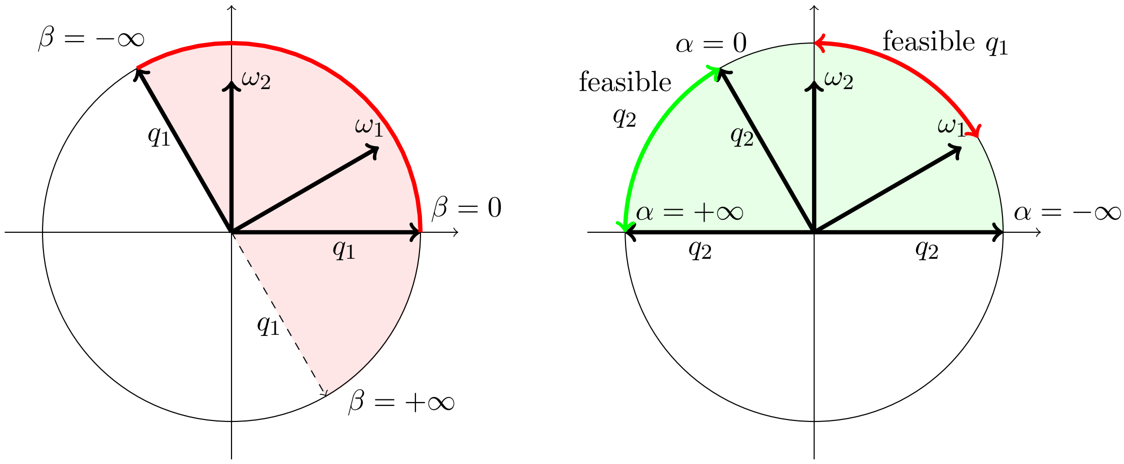
Although the formal proof is confined in the appendix, the intuition of the results can be obtained from the graphical representations reported in Figure 1 and Figure 2, depending on the two potential values of $\omega_{pq} \geq 0$ (Case I) and $\omega_{pq} < 0$ (Case II), respectively, under the sign normalization restriction discussed in Assumption 1 and sign restrictions of Assumption 2.

In Figure 1, left panel, we report the ω_1 and ω_2 vectors when the estimated $\omega_{pq} \geq 0$ (Case I), as well as all the possible q_1 vectors generating strictly negative values for the β parameter, as in an hypothetical demand curve. In the right panel, we also include the further restriction of positive values for the parameter α, consistent with a supply curve. The two sign restrictions, jointly, given the orthogonality condition regarding q_1 and q_2, implicitly impose a set restriction for α, in terms of the *feasible* q_2 vectors highlighted with the green arc in the right panel of Figure 1. The width of the set, as discussed in Proposition 5, depends on the estimable elements on the covariance matrix of the observable variables $\varOmega$, or equivalently, on the inverse of its Cholesky decomposition $\varOmega_{1,tr}^{-1}$.

Similarly, in Figure 2, we discuss the set identification of α and β when $\omega_{pq} < 0$ (Case II). In this latter case, a joint analysis of the sign restrictions on α and β (as well as the sign normalization), provides an identified set for β, leaving instead α to be unrestricted (though positive). The identified set for β, in the right panel, is highlighted by the red arrow indicating all feasible values for the q_1 vector.

Corollary 1 (Point identification and OLS estimation of α and β)**.** *Given the bivariate model in Eq. (14), under Assumption 1:*
(Case I): if $\omega_{pq} \geq 0$ and we restrict $\beta = 0$, then $\alpha = \frac{\omega_{pq}}{\omega_p^2}$
(Case II): if $\omega_{pq} < 0$ and we restrict $\alpha = 0$, then $\beta = \frac{\omega_{pq}}{\omega_q^2}$.

Figure 2: Identification of α and β. Case II: $\omega_{pq} < 0$



Sign restriction: $\beta \leq 0$ Sign restrictions: $\beta \leq 0$ and $\alpha \geq 0$

Notes: Set identification of the parameter β (left panel) and joint set identification of α and β (right panel). The identified set, under the sign restriction consistent with a demand curve, i.e. $\beta < 0$, is represented by the red arc in both panels. In the right panel, the set identification of α under the further sign restriction consistent with a supply curve, i.e. $\alpha \geq 0$, is represented by the green arc. In both cases, the standard assumption of positive diagonal terms on A_0 is considered ($\sigma_\varepsilon > 0$ and $\sigma_\eta > 0$): in light red for the first equation and in light green for the second equation.

Proof. See the Appendix B. □

The previous Corollary 1 simply restates a standard result in econometrics. In fact, if we introduce a further restriction on one of the two structural parameters, then the identified set, depending on the observed ω_{pq}, reduces to a single point (point identification of the other parameter), that can be consistently obtained through the OLS estimator, as stated in the corollary.

III.2 Point and set identification in bivariate HSVARs

Consider the model in Eq.s (14)-(15), but with a clear evidence of a shift in the variances of the observable variables. According to the HSVAR model introduced in Section II, such shift is simply due to a structural change involving the variances of the structural shocks, leaving unaffected the structural relationships among the variables, captured by the two parameters α and β, i.e.

$$\Sigma_i \equiv \begin{pmatrix} \sigma^2_{\varepsilon,i} & 0 \\ 0 & \sigma^2_{\eta,i} \end{pmatrix}, \qquad i = \{1, 2\},$$

where $i = 1$ denotes the first regime (before the break), $i = 2$ indicates the second regime (after the break), while the A matrix remains as defined in Eq. (15). Similarly as before, we standardize with respect to the standard deviations of the structural shocks in the first regime, and define

$$A_0 \equiv \Sigma_1^{-1/2} A = \begin{pmatrix} 1/\sigma_{\varepsilon,1} & -\beta/\sigma_{\varepsilon,1} \\ -\alpha/\sigma_{\eta,1} & 1/\sigma_{\eta,1} \end{pmatrix} \quad \text{and} \quad \Lambda \equiv \Sigma_1^{-1} \Sigma_2 = \begin{pmatrix} \sigma^2_{\varepsilon,2}/\sigma^2_{\varepsilon,1} & 0 \\ 0 & \sigma^2_{\eta,2}/\sigma^2_{\eta,1} \end{pmatrix},$$

and, coherently with the definitions in Sections II and II.2, let $C \equiv A_0^{-1}$.

From a different perspective, we consider that the change involves only the second moments of the distribution of the observable variables in $Y_t = (p_t\,,\,q_t)'$, showing the two covariance matrices

$$\Omega_i \equiv \begin{pmatrix} \omega_{p,i}^2 & \omega_{pq,i} \\ \omega_{pq,i} & \omega_{q,i}^2 \end{pmatrix}, \qquad i = \{1, 2\}\,, \tag{19}$$

that are connected to the structural parameters through the non-linear system of equations (9)-(10). When the solution of the system with respect to the structural parameters is unique, the identification problem is clearly solved. As before, we define the lower triangular Cholesky factorization of Ω_i as $\Omega_{tr,i}$, $i = \{1, 2\}$, whose inverses are given by

$$\Omega_{i,tr}^{-1} = \begin{pmatrix} \frac{1}{\omega_{p,i}} & 0 \\ \\ -\frac{\omega_{pq,i}}{\omega_{p,i}^2}\gamma_i & \gamma_i \end{pmatrix} = (\omega_{1,i}\,,\,\omega_{2,i})\,, \qquad i = \{1, 2\}\,, \tag{20}$$

where the two (2×1) vectors $\omega_{1,i}$ and $\omega_{2,i}$ are the two columns of $\Omega_{tr,i}^{-1}$, and where

$$\gamma_i = \left(\omega_{q,i}^2 - \frac{\omega_{pq,i}^2}{\omega_{p,i}^2}\right)^{-1/2}, \qquad i = \{1, 2\}\,. \tag{21}$$

Based on the connections between the reduced-form and the structural-form parameters highlighted in Eq.s (9)-(10), the identification issue can be addressed by studying the solutions of the system of equations reported in Eq. (12). Theorem 3 shows that it can be addressed as an eigen-decomposition problem that, under the condition of distinct eigenvalues, proves the structural parameters contained in (C, Λ) (or, equivalently in A, Σ_1 and Σ_2) to be point identified, up to permutations and sign changes of the structural equations. However, as a matter of comparison, we first report the Rigobon's condition for identification in bivariate HSVARs.

Theorem 6 (Rigobon (2003) condition for point identification in bivariate HSVAR)**.** *Given the HSVAR model described in Eq. (14) with the two covariance matrices reported in Eq. (19), under Assumption 1, a necessary and sufficient condition for the uniqueness of the structural parameters* (C, Λ) *is that*

$$\Omega_1 \neq a\Omega_2 \tag{22}$$

for any scalar $a > 0$.

Proof. See the proof of Theorem 1 in Rigobon (2003) and the proof of our Theorem 7 in the Appendix B. □

The following theorem, instead, restates the Rigobon's condition (Proposition 1, page 780, or our Theorem 6) as the solution of the eigen-decomposition problem based on the observed covariance matrices in Eq. (19). Moreover, it extends the results to the case of lack of identification due to equivalent eigenvalues.

Theorem 7 (Point and set identification in bivariate HSVAR)**.** *Given the HSVAR model described in Eq. (14) with the two covariance matrices reported in Eq. (19), the structural* *parameters (C, Λ) are obtained through the eigen-decomposition problem discussed in Theorem 3. In particular, the two variances of the structural shocks contained in Λ are given by the two eigenvalues of $\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1\prime}$, i.e.*

$$\lambda_{1,2} = \frac{\omega_{p,1}^2\omega_{q,2}^2 + \omega_{p,2}^2\omega_{q,1}^2 - 2\omega_{pq,1}\omega_{pq,2} \pm \Delta}{2\left(\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2\right)} \tag{23}$$

with

$$\Delta = \left[\left(\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2\right)^2 + 4\left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right)\left(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}\right) \right]^{1/2}.$$

The associated unit eigenvectors q_1 and q_2 form the columns of the orthogonal matrix $Q = (q_1\, , \, q_2)$ such that $C = \Omega_{1,tr}Q$. Under Assumption 1, the necessary and sufficient condition for the uniqueness of the structural parameters (C, Λ) is that $\Delta \neq 0$. If, instead, $\Delta = 0$, then Rigobon's condition fails and the HSVAR will only be set identified according to the results of the previous Theorem 5.

Proof. See the Appendix B. □

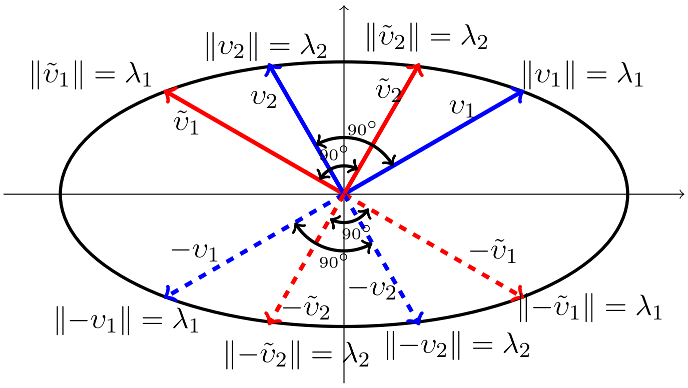
Theorem 7 provides analytical formula to calculate the structural parameters as a function of the eigenvalues and eigenvectors of observable matrices, i.e. the covariance matrices of the reduced form in the two regimes, Ω_1 and Ω_2. Furthermore, in proving the theorem, in the Appendix, we also show that the necessary and sufficient condition in Eq. (22), as expected, is equivalent to say that the two eigenvalues in Eq. (23) must be distinct (as postulated in Theorem 3) or, put differently, the shift in the variances of the structural shocks must be different.

However, when the quantity $\Delta = 0$, from Eq. (23) we have that the two eigenvalues λ_1 and λ_2 coincide and, according to Theorem 3, there will be infinite (not parallel) eigenvectors q_1 and q_2 and, as a consequence, the orthogonal $Q = (q_1\, , \, q_2)$ is not unique. The model, thus, is not point identified though we are in the presence of a structural break with distinct covariance matrices Ω_1 and Ω_2. Put differently, the information coming from the two different covariance matrices is not sufficient for point-identifying the structural parameters of the bivariate model.

We now move to the geometric interpretation of this result. Starting from Eq. (12), we easily obtain that $Q'\Omega_{1,tr}^{-1}\Omega_{2,tr}\Omega_{2,tr}'\Omega_{1,tr}^{-1\prime}Q = \Lambda$. Fixing the quantity $\Upsilon \equiv \Omega_{2,tr}'\Omega_{1,tr}^{-1\prime}Q$, then we have that $\Upsilon'\Upsilon = \Lambda$, or, equivalently, $\Upsilon\Lambda^{-1}\Upsilon' = I_n$, with I_n the $(n \times n)$ identity matrix. Interestingly, the columns of Υ, obtained as a linear transformation of the columns of Q, maintain the orthogonality condition, although their length is no longer unity, but given by the elements on the main diagonal of Λ. Coming back to the bivariate case, it is easy to remark that the equation $\Upsilon\Lambda^{-1}\Upsilon' = I_n$ is the representation of two orthogonal vectors, of length $\|\upsilon_1\| = \lambda_1$ and $\|\upsilon_2\| = \lambda_2$, belonging to an ellipse of equation $\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} = 1$, as shown in Figure 3.

Once we know λ_1 and λ_2, being the two eigenvalues of the eigen-decomposition highlighted in Theorem 3, there will be just two pairs of orthogonal vectors (other than their opposite), $(\upsilon_1\, , \, \upsilon_2)$ and $(\tilde{\upsilon}_1\, , \, \tilde{\upsilon}_2)$, having λ_1 and λ_2 as their length, shown in blue and in red, respectively, in Figure 3. Starting from these four pairs of vectors, using the definition

Figure 3: Identification of a bivariate HSVAR



Notes: Representation of the ellipse of equation $\frac{x^2}{\lambda_1} + \frac{y^2}{\lambda_2} = 1$, where λ_1 and λ_2 are the eigenvalues of $\Omega_{1,tr}^{-1}\Omega_2(\Omega_{1,tr}^{-1})'$. In blue the pairs of orthogonal vectors $(\upsilon_1\,,\,\upsilon_2)$ and $(-\upsilon_1\,,\,-\upsilon_2)$. In red the other two pairs of orthogonal vectors $(\tilde{\upsilon}_1\,,\,\tilde{\upsilon}_2)$ and $(-\tilde{\upsilon}_1\,,\,-\tilde{\upsilon}_2)$. The vectors $\upsilon_1, \tilde{\upsilon}_1, -\upsilon_1, -\tilde{\upsilon}_1$ have length exactly equal to λ_1, and $\upsilon_2, \tilde{\upsilon}_2, -\upsilon_2, -\tilde{\upsilon}_2$ have length exactly equal to λ_2. The following orthogonality conditions hold: $(\upsilon_1 \perp \upsilon_2)$, $(\tilde{\upsilon}_1 \perp \tilde{\upsilon}_2)$, $(-\upsilon_1 \perp -\upsilon_2)$, $(-\tilde{\upsilon}_1 \perp -\tilde{\upsilon}_2)$.

of Υ, it is possible to obtain four values of Q simply by linearly transforming the columns of Υ by the known quantities $\Omega_{1,tr}$ and $\Omega_{2,tr}$, i.e. $Q = \Omega_{1,tr}'\Omega_{2,tr}^{-1\prime}\Upsilon$. Based on Assumption 1 (sign normalization), just two of the four Q matrices will be retained (upper-half or lower-half of the ellipse). Fixing a specific ordering of the eigenvalues, or, equivalently, fixing the permutation matrix $P \in \mathcal{P}(n)$, helps reducing to one single admissible Q, making the HSVAR point identified. The problem arises when $\lambda_1 = \lambda_2$. The ellipse will collapse into a circle and infinite orthogonal vectors will be potentially admissible. In this case, of course, the HSVAR will be no longer identified.

IV Set-identification due to Indistinguishable Volatility Shifts

In this section we extend the preliminary results reported in Section II.2 where, conditional on the reduced-form parameters, the identification issue was addressed as an eigen-decomposition problem. Specifically, let $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the eigenvalues of the eigen-decomposition problem in Eq. (12), where $C = A_0^{-1}$ collects impact responses, Q is the orthogonal matrix containing the eigenvectors, and Ω_i, $i = \{1, 2\}$, are the reduced-form covariance matrices with $\Omega_{i,tr}$ the related Cholesky lower triangular matrices. Theorem 3 shows that a necessary condition for point identification of the structural parameters is the absence of multiplicity in eigenvalues in Λ. In the case of multiple eigenvalues, in fact, the identification of C fails. In particular, if two (or more) eigenvalues are equal, say $\lambda_j = \lambda_{j+1}$, then the corresponding columns of Q – i.e. the eigenvectors associated to λ_j and λ_{j+1} – denoted by q_j and q_{j+1}, are not unique. In fact they represent a basis for the two-dimensional vector space in $\mathbb{R}^n$, but any other couple of orthogonal unitary vector belonging to such a space could be an acceptable candidate to enter in the Q matrix. The matrix Q, thus, will not be a singleton in $\mathcal{O}(n)$ anymore, but will be a set of admissible orthogonal matrices solving the eigen-decomposition problem in Eq. (12).

More suitable notations and formalization are thus necessary. We start by formalizing the eigen-decomposition problem, with the possibility of multiple eigenvalues.

Definition 1 (Eigenspace of multiple eigenvalues)**.** Let the eigen-decomposition problem in Eq. (12) be characterized by the following eigenvalues

$$\lambda_1 \neq \ldots \neq \lambda_k$$

where the generic i-th distinct eigenvalue has algebraic multiplicity equal to m_i, i.e. $g(\lambda_i) = m_i$, $i = 1, \ldots, k$, with $\sum_{i=1}^{k} m_i = n$. Let $Q(\lambda_i)$ be the eigenspace associated to the i-th eigenvalue λ_i, i.e.

$$Q(\lambda_i) = \Big(\operatorname{span}\big(q_1^i, \ldots, q_{m_i}^i\big) \cap \mathcal{S}^{n-1} \Big) \subset \mathbb{R}^n \tag{24}$$

where $q_1^i, \ldots, q_{m_i}^i$ are linearly independent (not unique) eigenvectors associated to λ_i with $\mathcal{S}^{n-1}$ being the unit sphere in $\mathbb{R}^n$. Moreover, given the result in Lemma 3 in Appendix A, $\dim\big(Q(\lambda_i)\big) = m_i$. $\square$

According to Definition 1, let $Q_\lambda = Q(\lambda_1) \times \cdots \times Q(\lambda_k)$. It is possible to introduce the set of all admissible matrices Q as follows

$$\mathcal{Q}(\phi) = \Big\{ \big(q_1, q_2, \ldots, q_n\big) \in Q_\lambda \Big\}. \tag{25}$$

As in the case of multiplicities $\mathcal{Q}(\phi)$ is not a singleton in $\mathcal{O}(n)$, one could think of imposing restrictions, likewise it is traditionally done in SVARs. This will be the topic of the next section.

IV.1 Normalization, equality and sign restrictions

One of the characteristics of HSVARs is that the identification of the parameters is obtained from a statistical point of view, without imposing restrictions on the parameters. However, we have seen in Section II.2 that normalization restrictions are important and play a relevant role. Moreover, we will see that in some cases, imposing equality or sign restrictions can be interesting to improve the results obtained through HSVARs, especially when some of the assumptions in Theorems 2 to 4 are no longer valid, as the presence of multiplicities. In this section we discuss normalization restrictions first, then we move to the equality restrictions before concluding with sign restrictions.

Normalization restrictions

The normalization issue has been largely debated in econometrics. Specifically for SVAR models, we refer to Waggoner and Zha (2003) and, more recently, to Hamilton, Waggoner, and Zha (2007). They show that a poor normalization rule can invalidate statistical inference on the parameters. In our setup, the first normalization restriction consists in imposing the covariance matrix of the structural shocks to be the identity matrix in the first regime, i.e. $E(\varepsilon_t \varepsilon_t') = I_n$, and, as a consequence, the Λ matrix in the second regime, i.e. $E(\varepsilon_t \varepsilon_t') = \Lambda$, as already introduced in Eq. (2). However, Theorem 1 states that it is not sufficient because of the presence of possible permutation matrices changing the order of the elements of Λ. The first normalization rule, thus, is completed by fixing a particular permutation matrix $P \in \mathcal{P}(n)$ maintaining the order of the elements in Λ fixed, as specified by the researcher.

Moreover, we have seen in Theorem 1 that a further normalization rule is needed in order to fix the sign of the columns of $C \equiv A_0^{-1}$. We summarize the set of normalization restrictions in the following general definition.

Definition 2. (Normalization) A normalization rule can be characterized by a set $N \subset \mathbb{P}^S$ such that for any structural parameter point $\theta = (A_0, A_+, \varLambda) \in \mathbb{P}^S$, there exists a unique permutation matrix $P \in \mathcal{P}(n)$ and a unique diagonal matrix $S \in \mathcal{D}(n)$, with +1 and -1 along the diagonal, such that $(PSA_0, PSA_+, P\varLambda P') \in N$. □

Secondly, concerning the ordering of the elements in the diagonal matrix $\varLambda$, we assume

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n. \tag{26}$$

It is important to stress that all the results developed in the paper can be rephrased in terms of different normalization rules coherent with Definition 2.

Equality restrictions

The standard approach to address the identification issue in SVARs is to impose equality restrictions on the structural parameters or on particular linear and non-linear functions of them. Although not common in the literature of heteroskedastic SVARs *à la* Lanne and Lütkepohl (2008), we do not preclude this possibility and allow for possible equality and sign restrictions. We first consider the former, while the latter will be presented in the next section.

Giacomini and Kitagawa (2021) and Arias, Rubio-Ramírez, and Waggoner (2018), in the context of SVARs, stress that imposing constraints on the structural parameters, or on suitable functions of them, such as on the impulse responses, corresponds to restrict the columns of the orthogonal matrix $Q \in \mathcal{Q}(\phi)$, mapping the reduced-form parameters into the structural ones. Here below, we show that it also happens in the context of HSVARs. In fact, as shown in Eq. (12), the structural-form parameters can be defined as the product of the orthogonal matrix Q and quantities coming from the reduced form. As this latter elements are unrestricted, imposing restrictions is equivalent to constrain the columns of Q. The set of equality restrictions we consider are as follows:

$$((i,j)\text{-th element of } A_0^{-1}) = 0 \iff (e_i'\varOmega_{1,tr})q_j = 0, \tag{27}$$

$$((i,j)\text{-th element of } A_0) = 0 \iff (\varOmega_{1,tr}^{-1}e_j)'q_i = 0, \tag{28}$$

$$((i,j)\text{-th element of } A_l) = 0 \iff (\varOmega_{1,tr}^{-1}B_l e_j)'q_i = 0, \tag{29}$$

$$((i,j)\text{-th element of } CIR^{\infty}) = 0 \iff \left[e_i' \sum_{h=0}^{\infty} C_h(B)\varOmega_{1,tr}\right] q_j = 0, \tag{30}$$

where e_i is the i-th column of the identity matrix I_n. A compact notation for the restrictions can be given by

$$\mathbf{F}(\phi, Q) \equiv \begin{pmatrix} F_1(\phi)q_1 \\ F_2(\phi)q_2 \\ \vdots \\ F_n(\phi)q_n \end{pmatrix} = \mathbf{0} \tag{31}$$

where $F_i(\phi)$, of dimension $f_i \times n$, depends on the reduced-form parameters $\phi = (B, \varOmega_1, \varOmega_2)$ only, while q_i is the i-th column of Q. The total number of restrictions characterizing the HSVAR is given by $f = f_1 + \cdots + f_n$.

Focusing our attention on the i-th eigenspace as in the eigen-decomposition in Definition 1, let the set of restrictions on the vectors $\left(q_1^i, \ldots, , q_{m_i}^i\right) \in Q(\lambda_i)$ be contained in the $f^i \times n$ matrix $\mathbf{F}^i(\phi, Q)$, with f^i denoting the total number of restrictions on the vectors $\left(q_1^i, \ldots, , q_{m_i}^i\right)$. Moreover, let the f_j^i restrictions on the j-th vector q_j^i be defined as $F_j^i q_j^i = 0$. This allows us to introduce the following definition:

Definition 3 (Non redundant restrictions)**.** Given reduced-form parameter $\phi = (B, \Omega_1, \Omega_2)$, let the HSVAR be characterized by the eigen-decomposition in Definition 1. Moreover, let the m_i vectors $\left(q_1^i, \ldots, , q_{m_i}^i\right) \in Q(\lambda_i)$, $i = 1, \ldots, k$, be characterized by zero restrictions of the form $F_j^i(\phi) q_j^i = 0$. Such identifying restrictions are *non redundant* if, for $j = 1, \ldots, m_i$, the orthogonal vectors $\left(q_1^i, \ldots, , q_{j-1}^i\right)$ are linearly independent of the row vectors of $F_j^i(\phi)$. □

Bacchiocchi and Kitagawa (2021) introduced first this definition of non redundant restrictions to complement the result in Theorem 7 in Rubio-Ramírez, Waggoner, and Zha (2010) on the identification of SVARs. In the same way, we will use it for developing conditions for point identification in our HSVARs.

Sign restrictions

Uhlig (2005) proposes sign restrictions to impulse responses in order to obtain identified sets rather than point identification. Giacomini and Kitagawa (2021) and Arias, Rubio-Ramírez, and Waggoner (2018), instead, combine sign and zero restrictions to tighten the impulse response identified sets.

As for the equality restrictions, sign restrictions can be seen as constraints on the columns of the Q matrix. Suppose to impose a set of $s_{h,i}$ restrictions on the impulse responses to the i-th shock at the h-th horizon. We can write the sign restrictions as

$$S_{h,i}(\phi) q_i \geq \mathbf{0} \tag{32}$$

where, given the definition of the impulse response provided in Eq. (6), $S_{h,i} \equiv D_{h,i}\, C_h(B) \Omega_{1,tr}$ is a $s_{h,i} \times n$ matrix with $D_{h,i}$, of dimension $s_{h,1} \times n$, a selection matrix made of 1 and -1 elements indicating the restricted impulse responses. In a similar way, one can define restrictions directly on the structural parameters of the HSVAR. If we define the matrix S_i by stacking all the $S_{h,i}$ matrices involving sign restrictions on q_i at different horizons, the set of all sign restrictions due to the i-th shock can be defined as

$$S_i(\phi) q_i \geq \mathbf{0}. \tag{33}$$

A compact notation for all the sign restrictions can be defined by

$$\mathbf{S}(\phi, Q) \geq \mathbf{0}. \tag{34}$$

Admissible structural parameters and identified set

Based on the parametrization of the model and the set of all possible restrictions considered above, it is now possible to formally define when a point in the parametric space can be indicated as admissible.

Definition 4 (Admissible parameters)**.** A structural parameter point (A_0, A_+, Λ) is said admissible if it satisfies the normalization restrictions of Definition 2, the equality restrictions in Eq. (31) and the sign restrictions in Eq. (34). Given the set of reduced-form parameters $\phi \in \Phi$, the set of admissible parameters can be defined as

$$\mathcal{A}^r(\phi) \equiv \Big\{ \big(A_0, A_+, \Lambda\big) = \big(Q'\Omega_{1,tr}^{-1}, Q'\Omega_{1,tr}^{-1}B, \Lambda\big) \in N \,\Big|\, Q \in \mathcal{Q}(\phi),\ \mathbf{F}(\phi, Q) = \mathbf{0},\ \mathbf{S}(\phi, Q) \geq \mathbf{0} \Big\}.$$

□

At the same time, it is interesting to focus on the set of all the admissible matrices Q. We thus provide the following definition.

Definition 5 (Admissible Q matrices)**.** An orthogonal matrix Q is said admissible if, conditional on the reduced-form parameters, it satisfies the normalization restrictions of Definition 2, the equality restrictions in Eq. (31) and the sign restrictions in Eq. (34). The set of all admissible Q matrices is defined as

$$\mathcal{Q}(\phi\,|F, S) \equiv \Big\{ Q \in \mathcal{Q}(\phi) \,\Big|\, \big(A_0, A_+, \Lambda\big) \in \mathcal{A}^r(\phi) \Big\}.$$

□

Finally, given that the attention could not be limited to the structural parameters but on transformations of them, like impulse response functions, it is also important to define the so called identified set.

Definition 6 (Identified set)**.** Given the set of admissible Q matrices $\mathcal{Q}(\phi\,|F, S)$ according to Definition 5, an identified set is defined as

$$IS(\phi\,|F, S) \equiv \Big\{ \eta\big(\phi, Q\big) \,\Big|\, Q \in \mathcal{Q}(\phi\,|F, S) \Big\}.$$

with $\eta(\phi, Q)$ being the transformation of the structural parameters one is interested in, defined as

$$\eta(\phi, Q) = IR^h_{gj} = e'_g C_h(B)\Omega_{1,tr} Q e_j \equiv c'_{gh}(\phi)\, q_j$$

where IR^h_{gj} is the (g,j)-th element of IR^h (or, equivalently, IR^∞ or CIR^∞) and $c'_{gh}(\phi)$ is the g-th row of $C_h(B)\Omega_{1,tr}$. □

IV.2 Point-identification in HSVARs with Indistinguishable Volatility Shifts

As discussed in the previous sections, point identification in HSVAR can be achieved only if the eigen-decomposition problem in Eq. (12) is characterized by n distinct eigenvalues. This feature does correspond to non proportional shifts in the variances of the structural shocks among the two regimes.

If this is not the case, or, practically speaking, we do not have credible evidence on structural breaks on the second moments of some of the observable variables in our HSVAR, point identification can still be reached by combining heteroskedasticity with zero restrictions previously defined. The next theorem formalizes this intuition.

Theorem 8. *Consider an HSVAR characterized by the eigenvalues and eigenspaces as in Definition 1 and by the admissible parameters as in Definition 4. The structural parameter* $(A_0, A_+, \Lambda) \in \mathcal{A}^r(\phi)$ *is point identified if and only if, for each* λ_i*,* $i = 1, \ldots, k$*,*

$Q(\lambda_i) = \left(q_1^i, \ldots, q_{m_i}^i\right)$, *the unit-length vector* q_j^i *is subject to* $f_j^i = m_i - j$ *non-redundant zero restrictions, for* $j = 1, \ldots, m_i$.

Proof. See the Appendix A. $\square$

Remark 2. The previous theorem generalizes two important results in the literature of SVAR models. Firstly, when all the eigenvalues are distinct, then $k = n$ and the algebraic multiplicity $m_i = 1$, $i = 1, \ldots, n$. As a consequence, $Q(\lambda_i) = \left(q^i\right)$, and no zero restriction is needed for point identification, being $f^i = 1 - 1 = 0$, as originally introduced by Lanne and Lütkepohl (2008), and restated in the previous Theorems 2 and 3. Secondly, if one has no credible believes about structural breaks on the second moments of the observable variables, or the shift produces perfectly proportional covariance matrices in the two regimes, i.e. $\Omega_2 = \lambda\Omega_1$ for any positive scalar λ (specifically, $\lambda = 1$ in the case of no breaks), then $\Omega = \Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime} = \Omega_{1,tr}^{-1}\sqrt{\lambda}\Omega_{1,tr}\sqrt{\lambda}\Omega_{1,tr}^{\prime}\Omega_{1,tr}^{-1\prime} = \lambda I_n$; there is just an eigenvalue whose associated eigenspace is the whole $\mathcal{S}^{n-1}$, i.e the unit sphere in $\mathbb{R}^n$. The condition in Theorem 8 reduces to the identification condition for global identification in Rubio-Ramírez, Waggoner, and Zha (2010) (Theorem 7), with the recent extensions proposed by Bacchiocchi and Kitagawa (2021).

Remark 3. Identification in HSVARs is essentially a statistical issue, in the sense that, once the information contained in the data in terms of the two volatility regimes allows to point identify all the structural parameters, the path of the impulse responses allows the researcher to identify *a posteriori* the shocks of interest. In this respect, if the eigenvalues do not present multiplicity, the only task will be to see which shocks produce impulse responses coherent with the economic theory and label these shocks accordingly. The same happens if, even in the case of multiplicity, the shocks of interest are those corresponding to the eigenvalues with no multiplicity, whose eigenvectors (uniquely identified) will constitute the columns of Q one is interested in. A problem could arise when, in the case of multiplicity, none of the already identified impulse responses are consistent with what expected from economic theory for the shocks of interest. In this case, the results of Theorem 8 can be of extreme interest as including zero restrictions allows to point identified such shocks that, thus, will be identified based on economic restrictions rather than on statistical basis. Importantly, the number of restrictions required can be much less that what is required for traditional SVARs, as some of the columns of the Q matrix have been already identified by using the heteroskedasticity contained in the data. In our view, this can be an important added value of Theorem 8.

Example 1 (Distinct eigenvalues)**.** Consider a HSVAR with three variables, ($n = 3$), and $k = 3$ distinct eigenvalues, $\lambda_1 \neq \lambda_2 \neq \lambda_3$. In this case, $m_i = 1$ for all $i = 1, .., 3$. Theorem 8 states that the HSVAR is point identified if and only if the unit vector q_j^i is subject to $f_j^i = m_i - j$ zero restrictions for $j = 1$ and $i = 1, ..., 3$. It follows that no restrictions are needed because each λ_i is associated with a unique eigenvector q^i. To sum up, with $n = 3$ and three distinct eigenvalues, no zero restrictions are needed to point identify the HSVAR model in statistical sense.

Example 2 (Eigenvalue multiplicity)**.** Consider a HSVAR with three variables ($n = 3$) and $k = 2$ distinct eigenvalues, $\lambda_1 > \lambda_2$, $\lambda_2 = \lambda_3$. In this case, the first eigenvalue is distinct from the others, hence $m_1 = 1$, while the second eigenvalue has multiplicity $m_2 = 2$. Theorem 8 implies that, as far as the first unique eigenvalue λ_1 is concerned, we do not need any restriction on q_1^1 (i.e. $f_1^1 = 1 - 1 = 0$). The second eigenvalue, λ_2, is associated with $m_2 = 2$ linearly independent, not unique, eigenvectors (i.e. q_1^2 and q_2^2). Writing Q as $\left[q_1^1\ q_1^2\ q_2^2\right]$ point identification is achieved with $f_1^2 = 2 - 1 = 1$ zero restriction on q_1^2 and $f_2^2 = 2 - 2 = 0$ restriction on q_2^2. To sum up, with $n = 3$ and two distinct eigenvalues, a single exclusion restriction is enough to point identify the HSVAR model in statistical sense.

IV.3 Set-identification in HSVARs with Indistinguishable Volatility Shifts

The results obtained in Sections II.2 and III allow to point identify all the columns of $Q \in \mathcal{O}(n)$ associated with eigenvalues without multiplicity. For all the other columns, they can be point identified according to the particular pattern of zero restrictions suggested by Theorem 8. Let λ_i be an eigenvalue with algebraic multiplicity $g(\lambda_i) = m_i$, in this section we consider restrictions that make the $(q_1^i, \ldots, q_{m_i}^i)$ columns of Q only set identified, being

$$f_j^i \leq m_i - j, \qquad j = 1, \ldots, m_i \tag{35}$$

with strict inequality for at least one $j = \{1, \ldots, m_i\}$.

Example 3 (Set identification of a HSVAR with multiple eigenvalues)**.** Consider a HSVAR model with three variables $y_t = (y_{1,t}, y_{2,t}, y_{2,t})'$. Let us assume that there are $k = 2$ distinct eigenvalues: $\lambda_1 > \lambda_2$ and $\lambda_2 = \lambda_3$. Suppose that plotting the impulse responses to the first shock, we observe a pattern that is consistent with an economically meaningful structural shock. Given that the second eigenvalue, λ_2, has multiplicity $m_2 = 2$, we can write the matrix Q as $\left[q_1^1\ q_1^2\ q_2^2\right]$. Without any zero restriction, $f_1^2 < 1$ the second and third column of Q are only set identified.

Example 4 (Point identification with one zero restriction)**.** Let us continue with Example 3 and assume that we impose a zero restriction so that a second structural shock can be identified (provided its effects are consistent with expectations from economic theory). For instance, we might add an exclusion restriction on $C \equiv A_0^{-1}$ such that the second shock does not affect the first variable on impact (i.e. $c_{12} = 0$). Theorem 8 implies that point identification can be achieved with $f_1^2 = 2 - 1 = 1$ restriction on q_1^2, $f_1^1 = 1 - 1 = 0$ and $f_2^2 = 2 - 2 = 0$ restrictions on q_1^1 and q_2^2 respectively, where $Q = \left[q_1^1\ q_1^2\ q_1^2\right]$.

As in many empirical applications, suppose we are interested in one single shock. According to what already said in the previous sections, this corresponds to the identification (set or point) of one single column of the Q matrix. Moreover, according to Remark 3, it is crucial to understand whether the point identified impulse responses obtained through the eigenvalues without multiplicity can be compatible with the shock of interest. If this is not the case, it is likely to suppose this latter to be associated with the eigenspace generated by λ_i, with multiplicity m_i. We first introduce a specific ordering of the shocks according to the identifying restrictions in Eq. (35), and then provide the conditions for the identified set to be convex.

Definition 7 (Ordering of variables)**.** The variables associated with the eigenvalue λ_i, with algebraic multiplicity m_i, are ordered according to the number of zero restrictions on $(q_1^i, \ldots, q_{m_i}^i)$, and specifically, such that they follow the relation

$$f_1^i \geq f_2^i \geq \ldots \geq f_{m_i}^i \geq 0. \tag{36}$$

In case of ties, the shock of interest, represented by the j^*-th column of $(q_1^i, \ldots, q_{m_i}^i)$, is ordered first. In other words, let $j^* = 1$ if no other column has a larger number of restrictions than $q_{j^*}^1$. If $j^* \geq 2$, then let the variables be ordered such that $f_{j^*-1}^i > f_{j^*}^i$.

The next theorems, based on Proposition 3 in Giacomini and Kitagawa (2021), provides sufficient conditions for the impulse response identified set $IS(\phi\,|F, S)$ to be convex. Precisely, we first consider the case of zero restrictions only, and then extend to the case of zero and sign restrictions.

Theorem 9 (Convexity of identified set under zero restrictions)**.** *Consider an HSVAR characterized by the eigenvaules and eigenspaces as in Definition 1 and by the admissible parameters as in Definition 4. Let λ_i be an eigenvalue of algebraic multiplicity $g(\lambda_i) = m_i$, with associated eigenspace $Q(\lambda_i)$ as in Eq. (24), containing $q_{j^*}^i$, the column of Q associated with the j^*-th structural shock (shock of interest). Moreover, let $r = \eta(\phi, Q) = c'_{lh}(\phi) q_{j^*}^i \in IS(\phi\,|F, S)$ be the impulse responses to the shock of interest. Finally, let the variables be ordered as in Definition 7.*

Then, the identified set for r is non empty and bounded for any $l \in \{1, \ldots, n\}$ and $h = 1, 2, \ldots$, ϕ-a.s. Moreover, a sufficient condition for the identified set to be convex is that any of the following exclusive conditions holds:

1. *$j^* = 1$ and $f_1^i < m_i - 1$;*
2. *$j^* \geq 2$ and $f_j^i < m_i - j$, for $j = 1, \ldots, (j^* - 1)$;*
3. *$j^* \geq 2$ and there exists $1 \leq k < (j^* - 1)$ such that $(q_1^i, \ldots, q_k^i)$ is exactly identified as in Theorem 8 and $f_j^i < m_i - j$, for $j = k + 1, \ldots, j^*$.*

Proof. See Appendix A. □

The previous theorem just consider zero restrictions on the vectors of $Q(\lambda_i)$. The following one, instead, also allows for sign restrictions, although these last can be imposed on the vector $q_{j^*}^i$ associated with the shock of interest.

Theorem 10 (Convexity of identified set under zero and sign restrictions)**.** *Consider an HSVAR as in Theorem 9, where, as before, $q_{j^*}^i$ is the column vector corresponding to the shock of interest, and let $q_{j^*}^i \in Q(\lambda_i)$, the eigenspace associated with the eigenvalue λ_i, of algebraic multiplicity $g(\lambda_i) = m_i$. Moreover, let the sign restrictions be imposed on the shock of interest, only.*

1. *Let the zero restrictions $F^i(\phi, Q) = 0$ satisfy one of the conditions (1) and (2) of Theorem 9. If there exists a unit length vector $q \in \mathbb{R}^n$ such that*

$$F_{j^*}^i(\phi)\, q = 0 \quad \textit{and} \quad \begin{pmatrix} S_{j^*}(\phi) \\ \sigma^{j^*\prime} \end{pmatrix} q > 0 \tag{37}$$

then the identified set is non empty and convex for every $l \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$

2. Let the zero restrictions $F^i(\phi, Q) = 0$ satisfy condition (3) of Theorem 9, and let $(q_1^i\phi, \ldots, q_k^i\phi)$ be the first k vectors that are exactly identified. If there exists a unit vector $q \in \mathbb{R}^n$ such that

$$\begin{pmatrix} F^i_{j^*}(\phi) \\ v_1^{i\prime} \\ \vdots \\ v_{(n-m_i)}^{i\prime} \\ q_1^{i\prime}(\phi) \\ \vdots \\ q_k^{i\prime}(\phi) \end{pmatrix} q = 0 \quad \text{and} \quad \begin{pmatrix} S_{j^*}(\phi) \\ \sigma^{j^*\prime} \end{pmatrix} q > 0 \tag{38}$$

where $(v_1^i, \ldots, v_{(n-m_i)}^i)$ is a basis for the space $Q^{\perp}(\lambda_i)$, then the identified set is non empty and convex for every $l \in \{1, \ldots, n\}$ and $h = 0, 1, 2, \ldots$

Proof. See Appendix A. □

Taken jointly, the results of Theorems 9 and 10 generalize Proposition 3 in Giacomini and Kitagawa (2021) to the case of a structural break on the variances of the shocks, with potential multiplicity on the associated eigenvalues.

On the other side, the two theorems provide important insights on the possibility to apply the standard identification-through-heteroskedasticity approach to the case in which some of the switches in the variances are the same. As we will see in the next section, these new results represent the foundations for developing an estimator for the bounds of the identified set and produce the related inference.

V Inference in Set-identified HSVARs: a Robust Bayes Approach

In this section we present a completely brand new approach to conduct inference on set-identified HSVARs, where the set identification comes from the fact that not all the shifts in the variances of the shocks are statistically different. We first describe on how to estimate the reduced-form parameters of the model and on how to check for indistinguishable variances. Then, if some of such variances are not significantly different each other, we introduce our Robust Bayes approach to conduct inference on the identified set of interest.

V.1 Estimating the reduced-form HVAR

We present three estimators for estimating the parameters of the HVAR: two frequentist ones that will be used for detecting the presence of multiple eigenvalues and a Bayesian one that is at the heart of our procedure for making inference in the case of set identification.

Let the $nm \times 1$ vector of parameters $\phi_B = \text{vec}\,(B)$ and the $n \times T$ matrices $Y = [y_1, y_2 \ldots, y_T]$ containing the data, and $U = [u_1\, u_1 \ldots, u_t]$ containing the error terms. We can define $y = \text{vec}\,(Y)$ and $u = \text{vec}\,(U)$. Now, the presence of volatility clusters allows to write

$$V(U) = \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix} \tag{39}$$

where $T_1 = T_B$ and $T_2 = T - T_B$. Given the initial observations $y_{-l+1}, \ldots, y_0$, the $m \times T$ matrix $X = [x_1, \ldots, x_t \ldots, x_T]$, with $x_t = (1, y'_{t-1}, \ldots y'_{t-l})'$.

Given these definitions, the reduced-form HVAR in Eq. (3) can be written as

$$y = (X' \otimes I_n)\phi_B + u \qquad \text{or} \qquad Y = BX + U. \tag{40}$$

These compact notations, as well as a suitable partitioning of y and X as follows

$$y = \begin{pmatrix} \underset{nT_1 \times 1}{y_1} \\ \underset{nT_2 \times 1}{y_2} \end{pmatrix} \qquad \text{and} \qquad X = \begin{pmatrix} \underset{m \times T_1}{X_1} & \underset{m \times T_2}{X_2} \end{pmatrix} \tag{41}$$

allow to define a feasible generalized least squares (GLS) estimator[2]

$$\hat{\phi}_{B,GLS} = \Big[\big(X_1 X_1' \otimes \hat{\Omega}_1^{-1}\big) + \big(X_2 X_2' \otimes \hat{\Omega}_2^{-1}\big)\Big]^{-1} \Big[\big(X_1 \otimes \hat{\Omega}_1^{-1}\big)y_1 + \big(X_2 \otimes \hat{\Omega}_2^{-1}\big)y_2\Big] \tag{42}$$

where $\hat{\Omega}_i$, $i = \{1, 2\}$, is the covariance matrix of the residuals when Eq. (40) is estimated with equationwise ordinary least squares in a first step.

Moreover, apart from a constant term, and conditional on the initial observations $y_{-l+1}, \ldots, y_0$, the reduced-form Gaussian likelihood function can be written as

$$\begin{aligned} L(Y|\phi_B, \Omega_1, \Omega_2) \propto |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \exp\Bigg\{ -\frac{1}{2}[y - (X' \otimes I_n)\phi_B]' \ldots \\ \ldots \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} [y - (X' \otimes I_n)\phi_B]\Bigg\} \end{aligned} \tag{43}$$

If the data generating process is Gaussian, maximizing $L(Y|\phi_B, \Omega_1, \Omega_2)$ with respect to the parameters $(\phi_B, \Omega_1, \Omega_2)$ gives the maximum likelihood (ML) estimators. Instead, if departures from gaussianity do arise, the resulting estimators are quasi-ML estimators.

Furthermore, combining the likelihood function in Eq. (43) with the following Normal and inverse Wishart priors for ϕ_B, Ω_1 and Ω_2

$$\begin{aligned} \phi_B &\sim \mathcal{N}(\mu_\phi, V_\phi) \\ \Omega_1 &\sim i\mathcal{W}(S_1, d_1) \\ \Omega_2 &\sim i\mathcal{W}(S_2, d_2) \end{aligned}$$

allows to obtain the following posterior distributions for ϕ_B, Ω_1 and Ω_2

$$\begin{aligned} P(\phi_B, \Omega_1, \Omega_2|Y) \propto & |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} |\Omega_1|^{-\frac{d_1+n+1}{2}} |\Omega_2|^{-\frac{d_2+n+1}{2}} \\ & \exp\Big\{ -\frac{1}{2}[y - (X' \otimes I_n)\phi_B]' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} [y - (X' \otimes I_n)\phi_B]\Big\} \\ & \exp\{ -\frac{1}{2}\mathrm{tr}\,[\Omega_1^{-1} S_1]\} \exp\{ -\frac{1}{2}\mathrm{tr}\,[\Omega_2^{-1} S_2]\} \end{aligned} \tag{44}$$

This joint distribution is not of a known form and drawing directly from it is very hard. However, given the conditional distributions derived in Appendix C.2 for ϕ_B given Ω_1 and Ω_2, and those of Ω_1 and Ω_2 given ϕ_B, we can explore the posterior joint distribution by

[2] Details are provided in Appendix C.1.

using a Gibbs sampler.

V.2 Inference on the identified set

Now suppose some of the eigenvalues obtained by the eigen-decomposition in Eq. (12) present potential multiplicity. This evidence, for example, could be statistically checked by the Lütkepohl et al. (2020) or Lewis (2021) tests. Once this evidence is "statistically confirmed", one natural way of proceeding is to impose such eigenvalues to be effectively equal. This choice, however, imposes implicitly restrictions on the covariance matrices of the reduced form. While ML estimator subject to constraints on the parameters is generally implementable, it is rather problematic in the specific case of imposing equality restrictions among the eigenvalues. In such particular case, in fact, firstly, imposing the restrictions makes the model no longer identified, and thus creating convergence problems of the algorithm maximizing the likelihood function and, secondly, it is technically difficult to impose restrictions on some parameters that are observationally equivalent to permutations, as highlighted in Theorem 1.

Our strategy, instead, is based on the following lemma, that, according to evidence of potential eigenvalue multiplicity, suggests imposing the multiplicities as the result of a minimization problem about the unrestricted and restricted covariance matrices of the reduced-form HVAR.

Lemma 1 (Similarities of positive-definite symmetric real matrices)**.** *Let Ω be a $n \times n$ symmetric and positive definite real matrix characterized by the eigen-decomposition $\Omega = Q\Lambda Q'$, with the eigenvalues contained in the diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and the associated eigenvectors contained in the $n \times n$ orthogonal matrix Q. Moreover, let $\tilde{\Omega} = Q\tilde{\Lambda}Q'$, where the diagonal matrix $\tilde{\Lambda}$ contains the first m elements fixed to a scalar $\tilde{\lambda}$, while the remaining $n - m$ are the corresponding eigenvalues in Λ.*

Then, according to the Frobenius metric, $\min_{\tilde{\lambda}} \left\| \Omega - \tilde{\Omega} \right\|_F^2$ is reached when

$$\tilde{\lambda} = \frac{1}{m} \sum_{h=1}^{m} \lambda_h. \tag{45}$$

Proof. See Appendix A. □

The previous lemma provides a theoretical ground for fixing the common eigenvalues, when they are not statistically distinct, such that the unrestricted and restricted reduced-form covariance matrices are as close as possible, according to a specific metric. The assumption considered in the previous lemma is that the matrix Q, containing the eigen-vectors, is common in the two matrices Ω and $\tilde{\Omega}$. This assumption is completely reasonable for our problem in that it states that for the eigenvalues without multiplicity the eigenvectors are common in Ω and $\tilde{\Omega}$. Those associated to the eigenvalue with multiplicity, say λ_i, being not identified, must simply lay on the sub-space $Q(\lambda_i)$, orthogonal to $Q^{\perp}(\lambda_i)$. In this respect, the eigenvectors $(q_1^i, \ldots, q_m^i)$ obtained through the eigen-decomposition of Ω share this feature and can be used also as eigenvectors of $\tilde{\Omega}$. This explains the common Q matrix used in Lemma 1 both for Ω and $\tilde{\Omega}$.

Let $\tilde{\pi}_\phi$ be a probability measure on the space Φ of reduced-form parameters. In order to obtain a prior distribution for ϕ we need to restrict the support of $\tilde{\pi}_\phi$ such that its elements satisfy the sign, normalization and equality restrictions, as well as the fact that they show eigenvalue multiplicities as in Eq. (25). In this respect, we define the prior distribution for the reduced-form parameters as follows

$$\pi_\phi = \frac{\tilde{\pi}_\phi \, \mathbb{1}\big\{\mathcal{Q}(\phi\,|F,S) \neq \emptyset\big\}}{\tilde{\pi}_\phi \left(\big\{\mathcal{Q}(\phi\,|F,S) \neq \emptyset\big\}\right)}$$

that, by construction, assigns probability one to the distribution of data that admits eigenvalue multiplicity and is consistent with the identifying restrictions. As the structural parameters are a function of $\big(\phi, Q\big) \in \varPhi \times \mathcal{O}\,(n)$, we define a joint prior for the two sets of parameters $\big(\phi, Q\big)$ as $\pi_{\phi,Q} = \pi_{Q|\phi}\,\pi_\phi$, where $\pi_{Q|\phi}$ is supported on $\mathcal{Q}(\phi\,|F,S) \in \mathcal{O}\,(n)$.

In the case of no multiplicity, sign and permutation normalizations allow to pin down just one admissible Q, and $\pi_{Q|\phi}$ becomes a degenerate distribution centered on such Q. In the case of multiplicity and zero restrictions satisfying the pattern in Eq. (35), instead, the HSVAR will be only set identified and the prior $\pi_{Q|\phi}$ has to be specified in order to obtain a posterior distribution for the structural parameters and impulse responses, as desired in standard Bayesian approach. Other than being a challenging task for applied economists to specify $\pi_{Q|\phi}$, it has been shown that the choice of such a prior, being never updated by the data, can have non-negligible impact on the posterior inference even asymptotically (Baumeister and Hamilton, 2015).

In order to fix this unpleasant issue, we use the robust Bayes inference proposed by Giacomini and Kitagawa (2021). This approach consists in fixing a single prior π_ϕ for the reduced-form parameters, but a set of priors for $\pi_{Q|\phi}$. Specifically, given π_ϕ, we define

$$\varPi_{Q|\phi} = \Big\{\pi_{Q|\phi} :\ \pi_{Q|\phi}\big(\mathcal{Q}(\phi\,|F,S)\big) = 1,\ \pi_\phi - \text{almost surely}\Big\}$$

as the set of arbitrary priors $\pi_{Q|\phi}$ assigning probability one to the set of admissible Q matrices. This strategy allows to obtain a class of posteriors for (ϕ, Q)

$$\varPi_{\phi,Q|Y} = \Big\{\pi_{\phi,Q|Y} = \pi_{Q|\phi}\pi_\phi \ :\ \pi_{Q|\phi} \in \varPi_{Q|\phi}\Big\}$$

and, as a consequence, for the impulse response of interest $r = r(\phi, Q)$

$$\varPi_{r|Y} = \Big\{\pi_{r|Y}(\,\cdot\,) = \pi_{\phi,Q|Y}\big(r(\phi,Q) \in \,\cdot\,\big) :\ \pi_{\phi,Q|Y} \in \varPi_{\phi,Q|Y}\Big\}$$

for an event $\{r \in G\}$, where G is a measurable subset in $\mathbb{R}$, and Y is the sample of observations.

According to Giacomini and Kitagawa (2021), the results of this procedure can be summarized by reporting the posterior mean bounds interval, that can be seen as an estimator for the identified set, and an associated robustified credible region measuring the uncertainty related to the former. In particular, if we define $\ell(\phi) = \inf\big\{r(\phi,Q) :\ Q \in \mathcal{Q}(\phi\,|F,S)\big\}$ and $u(\phi) = \sup\big\{r(\phi,Q) :\ Q \in \mathcal{Q}(\phi\,|F,S)\big\}$, the posterior mean bounds interval can be written as

$$\left[\int_\varPhi \ell(\phi)\mathrm{d}\phi_\phi\,;\ \int_\varPhi u(\phi)\mathrm{d}\phi_\phi\right].$$

The robustified credible region, instead, consists in an interval C_α for which the posterior probability is greater than or equal to α uniformly, i.e.

$$\pi_{r|Y}(C_\alpha) \geq \alpha.$$

Finally, Giacomini and Kitagawa (2021) prove consistency for the range of the posterior means and show that such an interval represents the shortest-width robustified credible region with credibility $\alpha \in (0, 1)$, asymptotically.

V.3 Computing posterior bounds

In this subsection we present an algorithm to be used in the case of two volatility regimes in the data with known break date. This last assumption is rather standard in the literature, being the break dates associated to well documented changes in the policy conduct or to financial crises.[3]

Algorithm 1. *Let $y_{-l+1}, \dots, y_0, y_1, \dots, y_T$ be a sample of observations characterized by a break in the volatility occurred at time T_B, that is known or exogenously determined. Fix a normalization rule N.*

(Step 1) *Estimate the HSVAR model through the ML estimator as in Eq. (43),[4] obtain the estimated $\hat{Q}$ and $\hat{\Lambda}$ and check for eigenvalue multiplicity (e.g. Lütkepohl, Meitz, Netšunajev, and Saikkonen, 2020, or Lewis, 2021). If there is no multiplicity, or the shock of interest can be attributed to a particular q_{j^*} that comes out from an eigenvalue without multiplicity, then such shock is point identified (apart from sign) and the inference on the IRFs is standard. Then STOP.*
If there are multiplicities and the shock of interest cannot be attributed to the already identified columns of Q, then consider equality and sign restrictions, $\boldsymbol{F}(\phi, Q)$ and $\boldsymbol{S}(\phi, Q)$, respectively, to identify the shock of interest associated to $q^i_{j^*} \in Q(\lambda_i)$; then move to Step 2.

(Step 2) *Specify a prior for the reduced-form parameters $\tilde{\pi}_\phi$ and estimate a Bayesian HVAR as suggested in Section V.1 and obtain draws from the posterior distribution of $\tilde{\pi}_{\phi|Y}$, the parameters of the reduced form of the HVAR.*

(Step 3) *Take one draw $\phi = (B, \Omega_1, \Omega_2)$ from the posterior distribution of $\tilde{\pi}_{\phi|Y}$. From this draw obtain the covariance matrices Ω_1 and Ω_2. Solve the eigen-decomposition in Eq. (12) and collect the eigenvalues in the matrix Λ and the eigenvectors in the matrix Q.*

(Step 4) *Extract from Q the basis of the space $Q(\lambda_i)$, whose columns are associated with the possible multiple eigenvalues, and define the matrix $\bar{Q}_{\lambda_i}$ containing the $n - m_i$ eigenvectors orthogonal to $Q(\lambda_i)$. If the zero restrictions meet the rank condition in Theorem 8, then $\mathcal{Q}(\phi\,|F, S)$ is non-empty and the point identified columns of $Q(\lambda_i)$, for the draw of ϕ, can be easily determined through Algorithm 1 in RWZ; then move to Step 5. If, instead, the zero restrictions do not meet the rank condition in Theorem 8, then the model is only set identified and, given the draw of ϕ, check whether $\mathcal{Q}(\phi\,|F, S)$ is empty or not by following the sub-routines below:*

(Step 4.1) *Let $z_1 \sim N(0, I_n)$ be a draw of an n-variate standard normal random variable. Let $\tilde{q}^1_i = M_1 z_1$ be the $n \times 1$ residual vector in the linear projection of z_1 onto an $n \times f^i_1$ regressor matrix $F^i_1(\phi)'$. For $k = 2, \dots m_i$, run*

[3] See, among many others, Lanne and Lütkepohl (2008), Boivin and Giannoni (2006), Angelini, Bacchiocchi, Caggiano, and Fanelli (2019), Rigobon (2003), Bacchiocchi (2017), andrea Carriero, Marcellino, and Tornese (2023). Moreover, Rigobon (2003) also shows consistency of the estimated parameters in the case of break date miss-specification.

[4] Or through the feasible GLS as in Eq. (42).

the following procedure sequentially: draw $z_k \sim N(0, I_n)$ *and compute* $\tilde{q}_k^i = M_k z_k$*, where* $M_k z_k$ *is the residual vector in the linear projection of* z_k *onto the* $n \times (f_k^i + n - m_i + k - 1)$ *matrix* $\left(F_k^i(\phi)', \bar{Q}_{\lambda_i}, \tilde{q}_1^i, \ldots, \tilde{q}_{k-1}^i\right)$*. The vectors* $\tilde{q}_1^i, \ldots, \tilde{q}_{m_i}^i$ *are mutually orthogonal, orthogonal to* $\bar{Q}_{\lambda_i}$*, and satisfy the equality restrictions.*

(Step 4.2) Given $\tilde{q}_1^i, \ldots, \tilde{q}_{m_i}^i$ *obtained in the previous step, define*

$$Q_{\lambda_i} = \left[\pm \frac{\tilde{q}_1^i}{\|\tilde{q}_1^i\|}, \ldots, \pm \frac{\tilde{q}_{m_i}^i}{\|\tilde{q}_{m_i}^i\|} \right],$$

where $\|\cdot\|$ *is the Euclidean metric in* $\mathbb{R}$*, then arrange the sign of each column of* Q_{λ_i} *according to the sign normalization as defined by* $S \in \mathcal{D}(n)$*. Based on the obtained* Q_{λ_i} *with appropriate sign normalization, form the* Q *matrix by collecting the columns in* $\bar{Q}_{\lambda_i}$ *and* Q_{λ_i} *according to the correct ordering determined by the permutation matrix* $P \in \mathcal{P}(n)$*.*

(Step 4.3) Check whether Q *obtained in (Step 4.2) is such that* $(A_0, A_+) = \left(PSQ'\Sigma_{1,tr}^{-1},\, PSQ'\Sigma_{1,tr}^{-1}B\right)$*, for appropriate* $S \in \mathcal{D}(n)$ *and* $P \in \mathcal{P}(n)$*, satisfies the sign restrictions* $S(\phi, Q) \geq 0$*. If so, retain this* Q *and proceed to (Step 5). Otherwise, repeat (Step 4.1) and (Step 4.2) a maximum of* L *times (e.g.* $L = 3000$*) or until* Q *is obtained satisfying* $S(\phi, Q) \geq 0$*. If none of the* L *draws of* Q *satisfies* $S(\phi, Q) \geq 0$*, approximate* $\mathcal{Q}(\phi \,|F, S)$ *as being empty and return to (Step 3) with the following draw of* ϕ*.*

(Step 5) Given ϕ *and* $Q = \left(\bar{Q}_{\lambda_i}, Q_{\lambda_i}\right)$*, with the correct ordering determined by* $P \in \mathcal{P}(n)$*, and correct sign normalization determined by* $S \in \mathcal{D}(n)$*, obtained in (Step 4), compute the lower and upper bounds of* $IS(\phi \,|F, S)$ *by solving the following constrained nonlinear optimization problem:*

$$\begin{aligned} &\ell(\phi) = \arg\min_{Q_{\lambda_i}} c_{gh}'(\phi) q_{j*}^i, \\ \text{s.t.} \quad & Q'Q = I_n, \;\; \boldsymbol{F}(\phi, Q) = 0 \\ & \left(PSQ'\Sigma_{1,tr}^{-1},\, PSQ'\Sigma_{1,tr}^{-1}B\right) \in N, \;\; \text{and} \;\; \boldsymbol{S}(\phi, Q) \geq 0 \end{aligned}$$

and $u(\phi) = \arg\max_{Q_{\lambda_i}} c_{gh}'(\phi) q_{j*}^i$ *under the same set of constraints. If the zero restrictions meet the rank condition in Theorem 8, then* $Q = \left(\bar{Q}_{\lambda_i}, Q_{\lambda_i}\right)$ *is a singleton and* $\ell(\phi) = u(\phi)$*.*

(Step 6) Repeat (Step 3) - (Step 5) M *times to obtain* $\left[\ell(\phi_m),\, u(\phi_m)\right]$*,* $m = 1, \ldots, M$*. Approximate the set of posterior means by the sample averages of* $\Big(\ell(\phi_m),\, m = 1, \ldots, M\Big)$ *and* $\Big(u(\phi_m),\, m = 1, \ldots, M\Big)$*.*

(Step 7) To obtain an approximation of the smallest robust credible region with credibility $\alpha \in (0, 1)$*, define* $d(\eta, \phi) = \max\{|\eta - \ell(\phi)|, |\eta - u(\phi)|\}$*, and let* $\tilde{z}_\alpha(\eta)$ *be the sample* α*-th quantile of* $\left(d(\eta, \phi_m) \,:\, m = 1, \ldots M\right)$*. An approximated smallest robust credible region for* η *is an interval centered at* $\arg\min_\eta \tilde{z}_\alpha(\eta)$ *with radius* $\min_\eta \tilde{z}_\alpha(\eta)$*.*

Some remarks about the algorithm are in order. The first one is about the prior for the two covariance matrices Ω_1 and Ω_2. As the aim of the analysis is to highlight the possible eigenvalue multiplicity, it would be preferable to use diffuse priors, like diagonal matrices with equal values on the main diagonal, that from one side are non-informative and on the other side consider all the eigenvalues to be equal and let the likelihood function to play the relevant role in this respect.

Second, the way the draws from the posterior distribution are obtained depends on the theoretical results of Section V.1. Using independent priors for Ω_1, Ω_2 and ϕ_B allows to develop a Gibbs sampler that is rather simple and permits to explore the joint posterior distribution in a very convenient way. Step 3 of our algorithm is based on this approach for generating the draws ϕ from the distribution $\tilde{\pi}_{\phi|Y}$. Any alternative way, however, can be performed without altering the other steps of the algorithm.

Third, checking for the emptiness of the identified set in the case of sign restrictions is performed in Step 4 by using linear projections starting from normal draws, as in Giacomini and Kitagawa (2021) and many other contributions in the Bayesian literature. As an alternative, we could use the QR decomposition as proposed by Arias, Rubio-Ramírez, and Waggoner (2018).

Fourth, for each of the draws consistent with the zero and sign restrictions, we consider as there were eigenvalue multiplicities, regardless whether it is effectively so, from a statistical point of view, for the draw ϕ. In this respect, the eigenvectors associated to the potential multiple eigenvalues act as a basis for the space of the not identified columns of Q. From one side, this way of proceeding is extremely conservative as it competely ignores the amount of information contained in all those draws where all the eigenvalues are substantially distinguished. From the other side, however, it avoids the consequences of a pre-testing step to apply to each draw to statistically check about eigenvalue multiplicity. Our inference, thus, is robust to eigenvalue multiplicity in the sense that we apply the robust Bayesian approach by Giacomini and Kitagawa (2021) to the set identified columns of Q associated to the suspected multiple eigenvalues.

Fifth, the constrained nonlinear optimization problem in Step 5 is less demanding than the one in Algorithm 1 by Giacomini and Kitagawa (2021), as the argument is not the entire matrix Q but just a subset of its columns. Even if the HVAR model is relatively large, the number of eigenvectors generating the subspace $Q(\lambda_i)$ is in general relatively small and we do not expect concerns about the convergence properties of the numerical optimization step. On the contrary, we could replace Step 5 by a new algorithm in the spirit of Algorithm 2 in Giacomini and Kitagawa (2021), where the constrained nonlinear optimization problem is substituted by iterating many times Step 4.1-Step 4.3 and approximate the interval $\left[\ell(\phi_m),\ u(\phi_m)\right]$ with the minimum and maximum values obtained in such iterations. If the number of iterations goes to infinity, such alternative bounds still provide a consistent estimator of the identified set.

Sixth, the algorithm works even in the case the zero restrictions allow to point identify the matrix Q. In this case, the set $\mathcal{Q}(\phi\,|F,S)$ is always non-empty, and the constrained nonlinear optimization problem simply returns $\ell(\phi) = u(\phi)$. The inference, then, becomes standard.

From a theoretical point of view, Giacomini and Kitagawa (2021) discuss the importance of convexity, continuity and differentiability of the identified set $IS(\phi\,|F,S)$ for the posterior means to have a valid frequentist interpretation. Obviously, the same has to be verified in our setup. If this is the case, they show that the set of posterior means is a consistent estimator of the true identified set and the robust credible region is an asymptotically valid

Table 1: Estimated eigenvalues and tests for identification through heteroskedasticity

Panel (a). Estimated eigenvalues			
$\hat{\lambda}_1$		2.092	(0.374)
$\hat{\lambda}_2$		0.345	(0.060)
$\hat{\lambda}_3$		0.175	(0.031)
Panel (b). Tests for identification through heteroskedasticity			
H_0	$H_r(\hat{\kappa}_1, \hat{\kappa}_2)$	Degrees of freedom (r)	p-value
$\lambda_1 = \lambda_2 = \lambda_3$	47.131	5	0.0000
$\lambda_1 = \lambda_2$	20.536	2	0.0000
$\lambda_2 = \lambda_3$	3.241	2	0.1980

Notes: Panel (a) shows the estimated eigenvalues, $\hat{\lambda}_j$ for $j = 1, 2, 3$ and their standard errors in brackets. Panel (b) shows the test for identification through heteroskedasticity of Lütkepohl, Meitz, Netšunajev, and Saikkonen (2020). $H_r(\hat{\kappa}_1, \hat{\kappa}_2)$ is the test statistics with $r - 1$ degrees of freedom, where $\hat{\kappa}_m$ for $m = 1, 2$ is an estimate of the kurtosis of reduced-form residuals in the m-th volatility regime. See Appendix D.

confidence set for the true identified set.

Concerning convexity, we have already proved it in the previous Theorems 9 and 10. About continuity and differentiability, since we use the same set of zero and sign restrictions as in Giacomini and Kitagawa (2021), extending their Proposition 4 and Proposition 5 to our setup is straightforward. Definitely, appropriate choice of zero and sign restrictions associated with mild regularity conditions on the coefficient matrices of such restrictions guarantee our results on HSVARs to have a valid frequentist interpretation, as for traditional SVARs.

VI Empirical Application

We apply our methodology to the SVAR model for the global crude oil market of Kilian (2009) that includes three variables: the percent change in global crude oil production ($\varDelta prod_t$), an index of global real economic activity (rea_t) and the logarithm of the real price of crude oil (rpo_t). Data are monthly and the sample period runs from January 1973 through December 2007.[5]

While Kilian (2009) identifies three structural shocks that drive the real price of crude oil using a simple recursive scheme, Lütkepohl et al. (2020), Lütkepohl and Netšunajev (2014) and Lütkepohl (2013) show that the same structural innovations can also be recovered by exploiting the existence of distinct volatility regimes. We set the lag order[6] of the VAR equal to 6 and follow Lütkepohl et al. (2020) that distinguish between two volatility regimes with – an exogenously determined – change point in October 1987.

Table 1(a) shows the estimated eigenvalues and their standard errors. Recall that the variances of structural shocks are normalized to unity before the break and hence estimates in Table 1(a) represent the change in variances from the first to the second volatility regime. We see that the volatility of the structural shock associated with the first eigenvalue is larger after the break, while the remaining structural shocks have relative variances lower than

[5]See Appendix E for details and additional results.

[6]The Akaike information criteria – in line with the choice of Lütkepohl et al. (2020), Lütkepohl and Netšunajev (2014) and Lütkepohl (2013) – suggests a VAR of order 3. Kilian (2009) relies on a VAR of order 24 to capture long price cycles of crude oil. The selected lag order is thus a compromise between these two approaches.

Table 2: Sign Restrictions on impact responses ($C \equiv A_0^{-1}$) in model $\mathcal{M}_2$

	Oil supply disruption	Positive aggregate demand shock	Positive oil-specific demand shock
$\Delta prod_t$	(-)	+	*
rea_t	-	(+)	*
rpo_t	+	+	(+)

Notes: " * " denotes that the sign of the impact response is unrestricted. Signs along the main diagonal are in brackets to highlight that these are not actual sign restrictions, but sign normalizations placed on $C \equiv A_0^{-1}$.

unity in the second regime.

Table 1(b) illustrates that the test for identification through heteroskedasticity of Lütkepohl et al. (2020) does not allow to reject the null hypothesis $H_0 : \lambda_2 = \lambda_3$. Eigenvalue multiplicity implies that standard identification through heteroskedasticity, presented in Theorem 2, fails. In fact, only one structural shock can be statistically identified relying on changes in volatility. If the identified shock cannot be given an interpretation that is consistent with economic theory, or if one wishes to identify other structural shocks, additional sign or exclusion restrictions are needed. In this case, Theorem 8 and its implementation in Algorithm 1 become extremely useful.

VI.1 Point and set identification of oil supply and demand shocks

We write the relationship between structural innovations, ε_t and reduced-form errors, $u_t = C\varepsilon_t$ as follows:

$$\begin{pmatrix} u_t^{\Delta prod} \\ u_t^{rea} \\ u_t^{rpo} \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} \varepsilon_t^{\Delta prod} \\ \varepsilon_t^{rea} \\ \varepsilon_t^{rpo} \end{pmatrix} \tag{46}$$

where C corresponds to the impact response matrix (A_0^{-1}). We consider four SVAR based on different identifying restrictions:

- $\mathcal{M}_0$, a recursively identified SVAR model with $c_{12} = c_{13} = c_{23} = 0$;
- $\mathcal{M}_1$, a standard HSVAR model identified exploiting changes in volatility and assuming distinct eigenvalues;
- $\mathcal{M}_2$, an HSVAR model that imposes eigenvalue multiplicity and exploits static and dynamic sign restrictions;
- $\mathcal{M}_3$, an HSVAR model that allows for eigenvalue multiplicity and imposes one exclusion restriction ($c_{21} = 0$).

Model $\mathcal{M}_0$ is the recursively identified SVAR model of Kilian (2009) used a as a benchmark against which we compare results from HSVAR models. Three exclusion restrictions – $c_{12} = c_{13} = c_{23} = 0$ – allow to point identify an oil supply shock and two demand shocks (i.e. aggregate and oil-specific demand shocks). Oil supply shocks represent innovations to the current physical availability of crude oil. Aggregate demand shocks capture unexpected changes of the demand for all industrial commodities driven by fluctuations in the global business cycle, while oil-specific demand shocks represent shifts in the precautionary demand for crude oil triggered by concerns about the future availability of supplies.

The HSVAR model $\mathcal{M}_1$ exploits changes in volatility for identifying structural shocks, without imposing additional exclusion or sign restrictions. We assume the eigenvalues to be all distinct, although results in Table 1(b) highlight the existence of eigenvalue multiplicity.

For this reason, in model $\mathcal{M}_2$ we impose the constraint $\lambda_2 = \lambda_3$ in Step 3 of Algorithm 1. With two distinct eigenvalues, we can point identify only one structural shock that, as will be shown in Section VI.2, yields impulse responses consistent with those associated with an oil-specific demand shock. The remaining structural shocks are set identified combining static and dynamic sign restrictions. See Table 2. We postulate that a negative oil supply shock lowers the real price of crude oil and depresses global real economic activity on impact. A positive aggregate demand shock is expected to raise oil price and production on impact. Notice that we also place sign normalizations on the main diagonal of $C \equiv A_0^{-1}$. Furthermore, we constraint the sign of the response of real crude oil price to oil supply disruptions to be positive for twelve months, starting from the impact response. These additional restrictions rule out models with the real price of crude oil decreasing below its starting level after a negative oil supply shock.

While specification $\mathcal{M}_2$ allows to point identify a single structural shock, because of eigenvalue multiplicity the HSVAR model remains set identified. In this case, Theorem 8 shows that point identification of the HSVAR model can be achieved with a single exclusion restriction. In model $\mathcal{M}_3$, we add one exclusion restriction on the impact response matrix: $c_{21} = 0$. This restriction implies that the first shock does not affect real economic activity within the same month. Compared to the recursively identified model, $\mathcal{M}_0$, when a volatility shift is exploited the identification scheme is less demanding. In fact in model $\mathcal{M}_3$ point identification is achieved – at least in statistical sense – combining the shift in the volatility of one of the structural shocks with a single zero restriction.

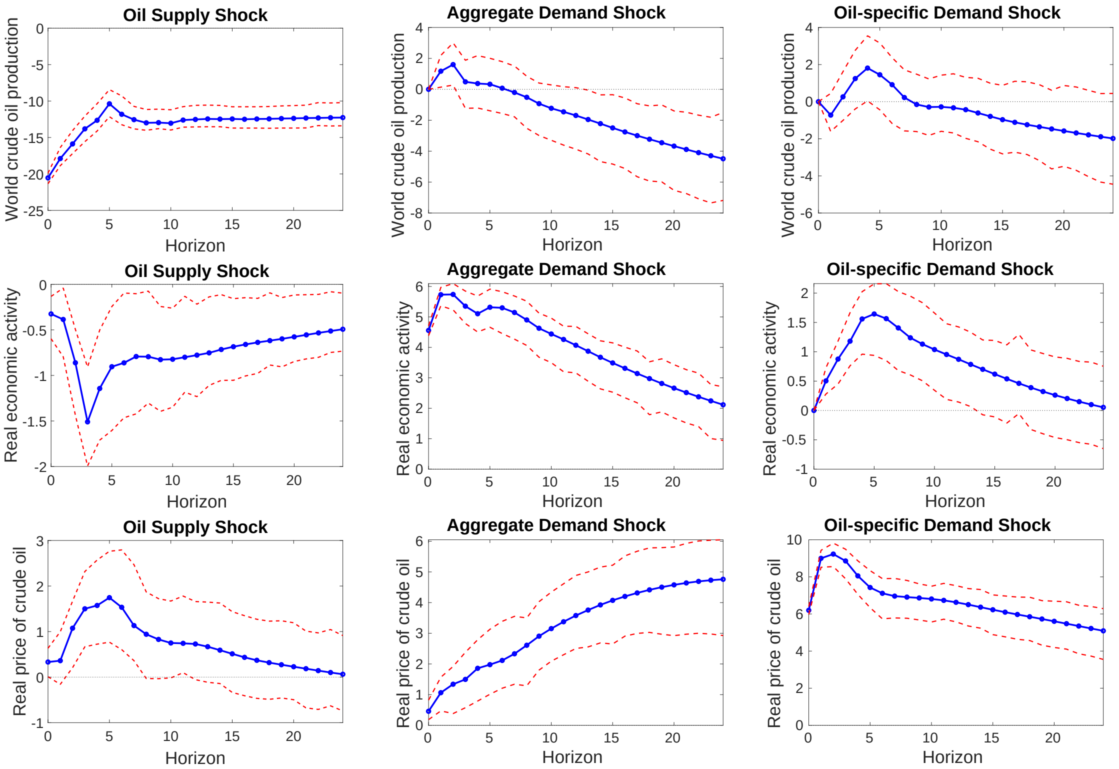
VI.2 Impulse response analysis

All specifications are estimated with Bayesian techniques drawing from the posterior of reduced-form parameters until we obtain 1000 realizations of the non-empty identified set. We estimate the whole set of structural impulse response functions over an horizon of 24 months. Notice that we report the implied response of world crude oil production obtained by cumulating that for $\Delta prod_t$. We focus on shocks that are expected to raise the real price of crude oil. **SIZE of SHOCKS?? ONE STD dev?**Therefore, in the case of supply shocks we plot the responses to a negative shock representing a disruption of crude oil supply.

Figure 4 shows impulse responses[7] for the recursively identified model of Kilian (2009), $\mathcal{M}_0$, along with the highest posterior density (HPD) region with credibility 68%. An oil supply shock causes an immediate and long-lasting decline in global oil production, a decrease in real economic activity and a transitory increase in the real price of crude oil that peaks six months after the shock. Notice that the 68% HPD region of the price response does not include zero only for the first eight months. A shock boosting aggregate demand causes a small temporary increase in global oil production and large and persistent increase in the index of real economic activity and in the price of crude oil. For the latter two responses the 68% HPD region never contains zero. An unexpected rise in oil-specific demand generates a long-lasting increase in the real price of crude oil and a temporary jump in the

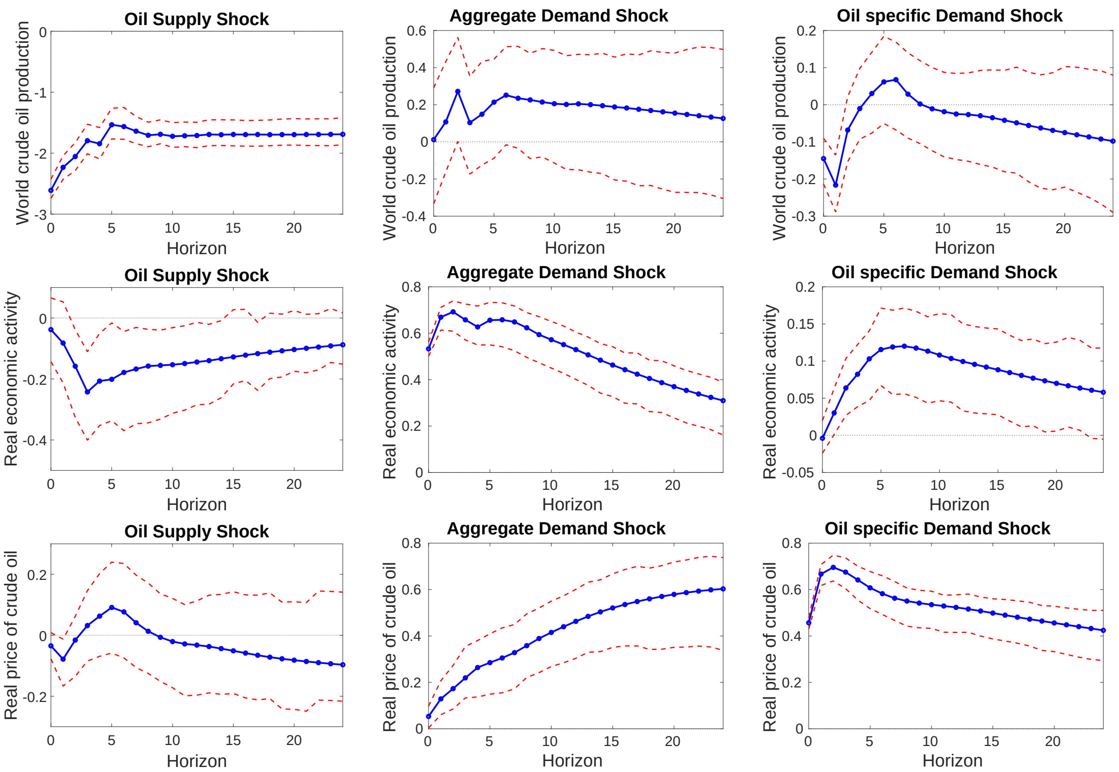
[7]As for the estimation, we rely on a noninformative improper Jeffreys' prior that allows to draw reduced-form parameters from a normal-inverse-Wishart posterior.

Figure 4: Impulse response functions $\mathcal{M}_0$



Notes: the blue line with dots represents the posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Recursive identification imposing $c_{12} = c_{13} = c_{23} = 0$. The model is point-identified

Figure 5: Impulse response functions $\mathcal{M}_1$

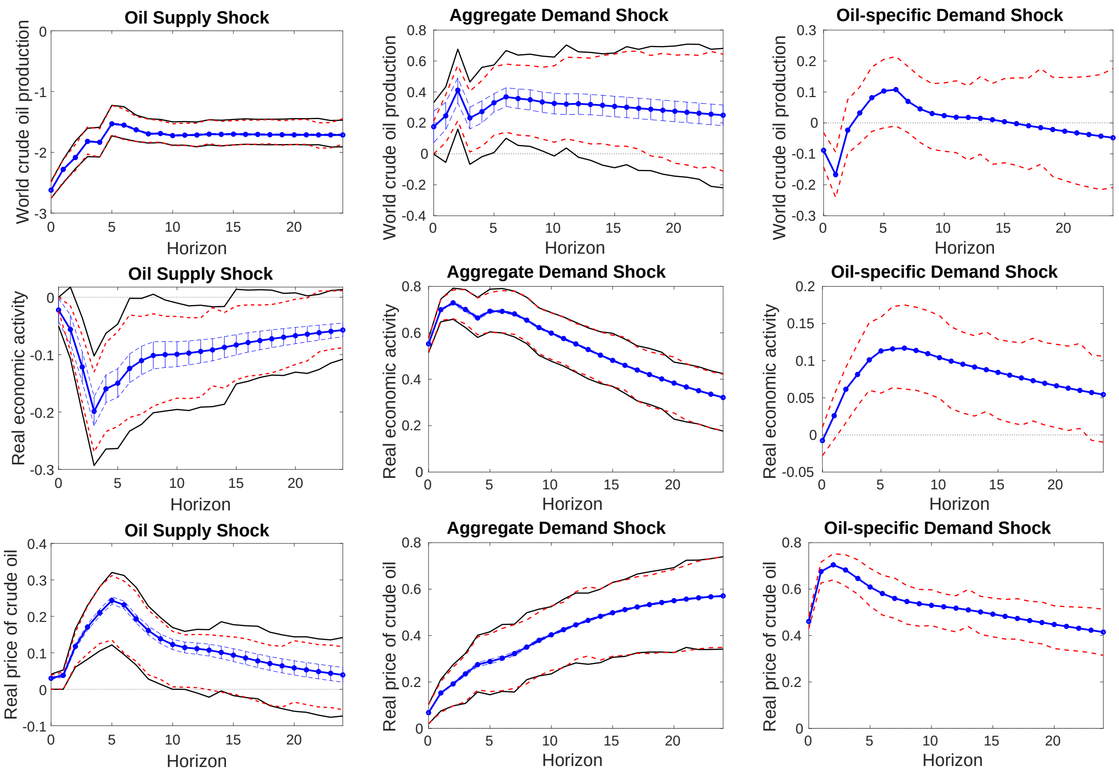


Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Identification is obtained via heteroskedasticity assuming distinct eigenvalues index of real economic activity. Lastly, an oil market demand shock causes a small and only transitory positive effect on global oil production. Notice that in this case the 68% HPD always includes zero.

Impulse responses from HSVAR models $\mathcal{M}_1$-$\mathcal{M}_3$ are displayed in figures 5-8. Since HSVAR models normalize structural residuals to have identity covariance matrix in the first regime, the scaling of these figures is not the same as that of figure 4.

Model $\mathcal{M}_1$ assumes that all eigenvalues are distinct and hence we estimate the reduced form of the model with the Gibbs sampler discussed in Section V.1. The last column of figure 5 shows the responses to the shock that is associated with the only distinct eigenvalue. Comparing the shape of these responses to that in figure 4, we see that they are consistent with those following an oil-specific demand shock. As for responses in the remaining columns, we observe that the 68% HPD region is generally wider than in model $\mathcal{M}_0$. Moreover, while impulse responses in the second column of figure 5 are consistent with those induced by an aggregate demand shock, we struggle to identify an oil supply shock. Focusing on the response of the real price of crude oil to an oil supply shock, we see that the 68% HPD region always contains zero. Similarly, the response of real economic activity to an oil supply disruption is very modest and less long-lived if compared to that in figure 4. All in all, these results highlight that changes in volatility alone, might not convey enough information to point identify all structural shocks.

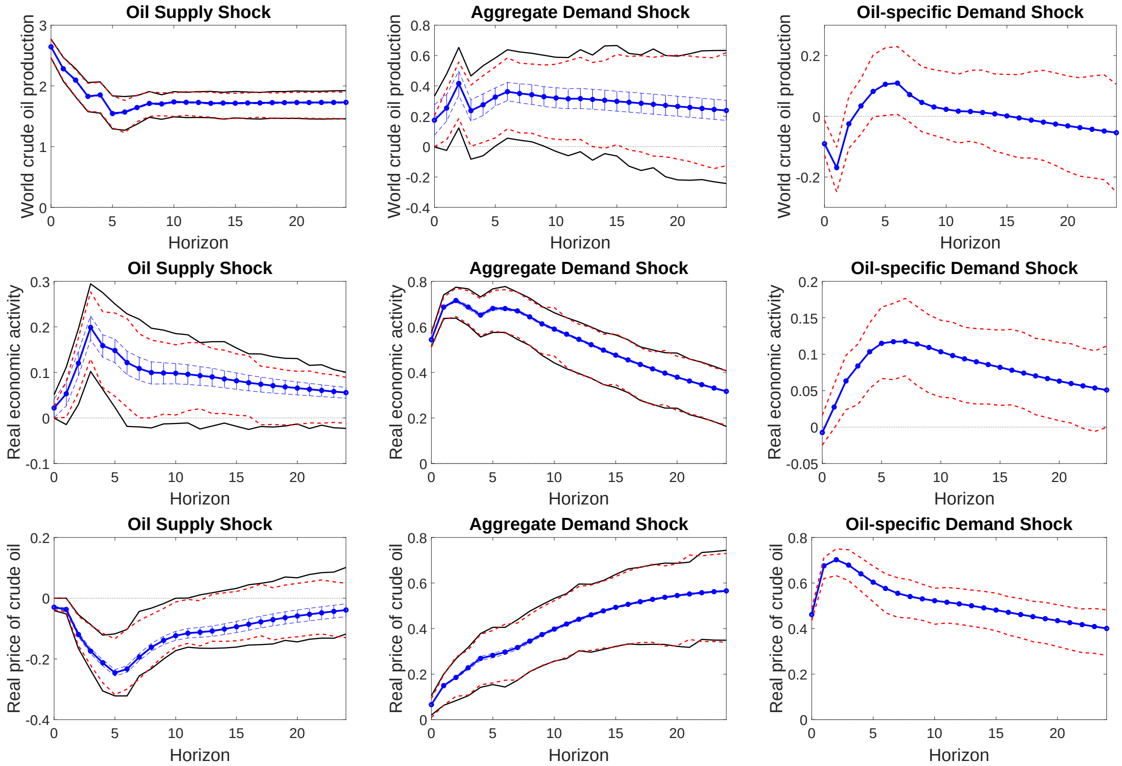
Figure 6: Impulse response functions $\mathcal{M}_2$



Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions.

Static and dynamic sign restrictions in model $\mathcal{M}_2$ allow to set identity supply and demand shocks that are expected to drive the real price of crude oil. Since the only distinct eigenvalue is associated with the oil-specific demand shock, sign restrictions are imposed

Figure 7: Impulse response functions $\mathcal{M}_2$ NO TEST



Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions.

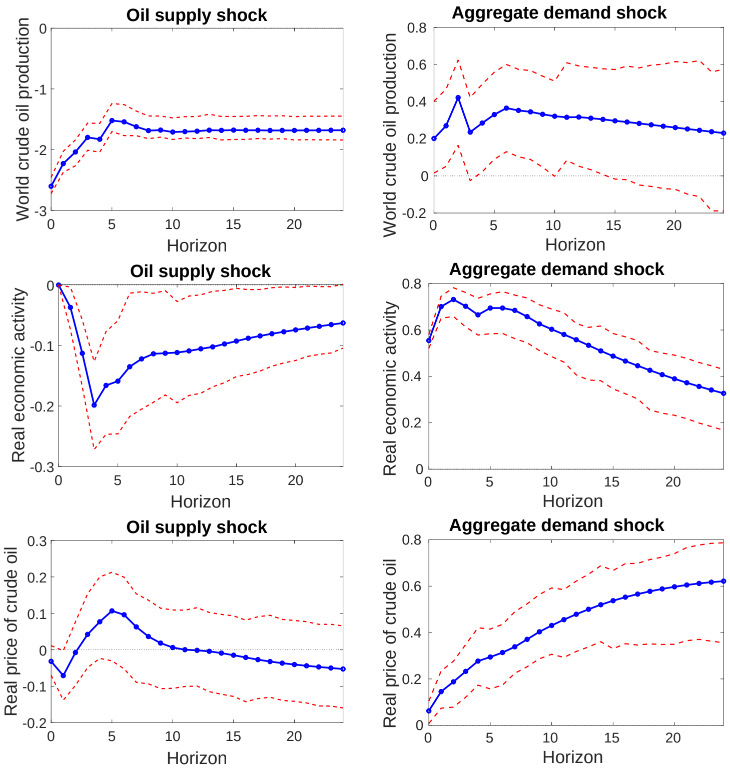
on the remaining columns of Q. The impulse responses appearing in the first two columns of figure 7 are those concerning those columns of Q and as such we also present the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% obtained with Algorithm 1.[8]

Imposing sign restrictions we recover oil supply and aggregated demand shock whose effects on the endogenous variables of the VAR are consistent with expectations from economic theory and previous analyses (see e.g. Kilian and Murphy, 2012). The shape of impulse responses in the first two columns of figure 7 are in line with those implied by the recursively identified model, $\mathcal{M}_0$. Also notice that the combination of volatility changes and sign restrictions delivers reasonable impulse responses even in the presence of multiple eigenvalues and using less sign restrictions than what is usually done in the literature. In fact, we leave one of the columns of Q completely unrestricted and exploit changes in volatility to point identify the corresponding shock. Contrary to the standard HSVAR, $\mathcal{M}_1$, oil supply shocks implied by $\mathcal{M}_2$ are associated with impulse responses that reasonably summarize the expected effects of such shocks on real economic activity and the real price of crude oil. In fact, we now observe a positive response of the price of crude oil with a peak after 6 months.

The width of the HPD and of the robust credible regions for both oil supply and aggregate demand shocks are similar. We can thus draw essentially the same conclusions using

[8]Our implementation is based on the constrained nonlinear optimization problem highlighted in Step 5 of Algorithm 1. We follow Giacomini and Kitagawa (2021) that mitigate possible convergence problems using five different starting values for the optimization problem in Step 5.

Figure 8: Impulse response functions $\mathcal{M}_3$



Notes: the blue line with dots represents the posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Identification through heteroskedasticity exploiting the fact that one eigenvalue is distinct from the others; moreover, we impose $c_{21} = 0$.

any of them.[9] Giacomini and Kitagawa (2021) propose a measure of the informativeness of the choice of an unrevisable prior for Q that compares the width of such regions. The fact that in our case such measure is generally small (at any horizon of the set identified impulse responses), indicates that the fraction of the credible region tightened by choosing a particular unrevisable prior is very modest.[10]

Model $\mathcal{M}_3$ illustrates that our methodology allows to point identify HSVAR models in the presence of multiple eigenvalues and that this can be achieved with less zero restriction than in the case of recursive identification. In $\mathcal{M}_3$ we impose that the first shock does not affect real economic activity within the same month. Interestingly, this restriction is consistent with the evidence in figures 4 and 7 where we see that the impact response of real economic activity to an oil supply shock is close to zero. As for figures 7 the highest posterior density and the robust credible region does contain the zero.

The zero restriction has the effect of tightening the width of the HPD regions compared to the standard HSVAR model, $\mathcal{M}_1$. However, focusing on the response of real price to supply shock, we see that also in this case the HPD region always contains the zero. Another

[9]Results based on an alternative implementation of Algorithm 1 are almost identical. In such implementation, we follow Giacomini and Kitagawa (2021) and substitute Step 5 with 10000 iterations of Step 4.1-Step 4.3. The interval $[\ell(\phi_m),\, u(\phi_m)]$ is then approximated by the minimum and maximum values over such iterations. This also confirms the convergence of the numerical algorithm in Step 5. See Appendix E.

[10]The informativeness of the prior with credibility α is defined as $\{1 - [\text{width highest posterior density}(\alpha)/\text{width robust credible region}(\alpha)]\}$ Such fraction is in the range 0.05-0.52 for the impact response to an oil supply shock (i.e. the largest fraction is that associated with the response of real economic activity) and in the range 0.03-0.37 for the impact response to an aggregate demand shock (i.e. the largest fraction is that associated with the response of world crude oil production). At horizon 12 such intervals become 0.03-0.21 and 0.06-0.22.

difference with $\mathcal{M}_1$ concerns the response of world crude oil production to an aggregate demand shock that – similarly to $\mathcal{M}_0$ and $\mathcal{M}_2$ – is positive with highest posterior density region that does not contain the zero up to horizon 12.

VI.3 Implications for structural parameters in oil market models

Two of the exclusion restrictions that $\mathcal{M}_0$ places on C (i.e. $c_{12} = c_{13} = 0$), imply that the short-run oil supply curve is vertical and hence perfectly inelastic. In the context of the present model, the impact price elasticity of oil supply is defined as the ratio of the impact response of global oil production triggered by an exogenous demand shock, relative to the impact response of the real price of crude oil, in response to the same demand shock.

While there is a consensus that the supply curve is relatively inelastic, assuming that the short-run supply elasticity is literally zero is just an approximation and several researchers have thus considered identification schemes that avoid such assumption (see e.g. Baumeister and Peersman, 2013; Baumeister and Hamilton, 2019; Kilian and Murphy, 2012, 2014; Lütkepohl and Netšunajev, 2014). HSVAR models $\mathcal{M}_1$–$\mathcal{M}_3$ rely on changes in volatility and do not place restrictions implying that the impact price elasticity of oil supply is exactly zero.

Estimates of the short-run supply elasticity implied by different models are shown in Table 3. Since in Equation (46) two different types of demand shocks affect the real price of oil, the impact price elasticity of oil supply can be defined either as $\eta_1 \equiv c_{13}/c_{33}$, or as $\eta_2 \equiv c_{12}/c_{32}$. The ratio $\eta_1 \equiv c_{13}/c_{33}$ is the short-run price elasticity of supply focusing on the oil-specific demand shock. Similarly, $\eta_2 \equiv c_{12}/c_{32}$ is the short-run price elasticity of supply focusing on the aggregate demand shock. In Table 3 we report the median value of η_1 and η_2 over 1000 draws that satisfy the restrictions imposed by a given model and the associated HPD region with credibility 68%.

As highlighted by Kilian and Murphy (2012, 2014), in the case of sign identified oil market models, the assumption that all draws satisfying the restrictions are equally likely gives rise to impulse responses associated with extremely large values of the short-run supply elasticity. For this reason, these authors propose an upper bound for η_i for $i = 1, 2$. Table 3 displays the percentage of draws for which $0 \leq \eta_i \leq 0.04$. While Kilian and Murphy (2012, 2014) suggest using 0.0258, we follow Zhou (2020) who relies on more recent empirical estimates of the price elasticity of oil supply and set the upper bound to 0.04.

Focusing on median elasticities and HPD regions, we see that HSVAR models do not provide sensible estimates of the elasticity of oil supply, thus confirming the findings in Kilian and Murphy (2012). For instance, we note that η_2 takes on negative values in the case of models $\mathcal{M}_1$-$\mathcal{M}_3$. This happens because while a sign normalization is imposed on c_{33}, we do not constraint the sign of c_{13} (i.e. the impact response of real economic activity to an oil specific demand shock). As for the HPD regions, these are generally very wide and all of them – with the exception of that for η_2 in the standard HSVAR model, $\mathcal{M}_1$ – contain zero.

The last column of Table 3 shows that the standard HSVAR model, $\mathcal{M}_1$, never recovers elasticity estimates in the suggested interval. On the contrary, $\mathcal{M}_2$ and $\mathcal{M}_3$ deliver a small percentage of draws satisfying the elasticity bound for both η_1 and η_2. Using a tighter elasticity bound of 0.0258 – as proposed by Kilian and Murphy (2012) – results in the last column of Table 3 change as follows: 100.0, 0.0, 2.1, 0.8. While these percentages are small, Kilian and Murphy (2012) report that only 0.26% (i.e. 80 out of 30,860) of their models

Table 3: Price elasticity of oil supply implied by different models

	$\eta_1 \equiv c_{12}/c_{32}$	68% $HPD(\eta_1)$		$\eta_2 \equiv c_{13}/c_{33}$	68% $HPD(\eta_2)$		$0 \leq \eta_i \leq 0.04$
$\mathcal{M}_0$	0.000	0.000	0.000	0.000	0.000	0.000	100.0
$\mathcal{M}_1$	0.225	-6.345	8.615	-0.316	-0.463	-0.195	0.0
$\mathcal{M}_2$	2.185	-3.756	3.765	-0.191	-0.321	-0.076	3.4
$\mathcal{M}_3$	2.396	-0.935	8.283	-0.305	-0.441	-0.160	1.0

Notes: η_i for $i = 1, 2$ is the median of the impact price elasticity of oil supply; HPD denotes the corresponding highest posterior density with credibility 68%. $0 \leq \eta_i \leq 0.04$ for $i = 1, 2$ is the percentage of draws – out of 1000 – with η_1 and η_2 in the interval.

satisfy the elasticity upper bound of 0.0258. Clearly, these percentages are not directly comparable for several reasons, including the fact that Kilian and Murphy (2012) set two additional sign restrictions on the impact effects of an oil demand shock. Nevertheless, imposing eigenvalue multiplicity seems to improve upon the standard HSVAR model that never recovers sensible values for the elasticity of oil supply.

A second structural parameter of interest is c_{23} that in Equation (46) captures the impact response of real economic activity to an oil-specific demand shock. Since such shock captures revision in expectations about the future availability of supplies, it affects real economic activity only through its impact on crude oil price. While we might expect c_{23} to be negative, large values are unlikely. The recursively identified model, $\mathcal{M}_0$, sets $c_{23} = 0$. This assumption is confirmed by HSVAR models $\mathcal{M}_1$-$\mathcal{M}_3$ that deliver median values of c_{23} that are negative and but very close to zero. Still, the percentage of impulse responses in the range (-1.5,0) is highest for $\mathcal{M}_2$, confirming that HSVAR models can be used to recover sensible structural parameters even in the presence of eigenvalue multiplicity if our methodology is used.

VII Conclusion

This paper deals with SVAR models with structural breaks, offering some new contributions. We first study the identification theory and propose a set of results for easily checking whether the model is globally identified. Second, we study the consequences on the impulse response functions of HSVARs that do not satisfy such identifying conditions. We deal with a SVAR model with heteroskedasticity, where non distinct changes in the variance shifts raise an identification issue. We solve the identification problem by imposing equality and sign restrictions and provide a methodology that helps giving a structural economic interpretation to the set or point identified shocks, by requiring fewer restrictions to be imposed. A way to do inference on the model both in case of point and set-identification is also proposed, as well as an empirical application of approach to the global crude oil market model. Some issues remain to be addressed by future research, such as extending the model to more than two volatility regimes and analysing the consequences of having an unknown break date.

References

andrea Carriero, M. Marcellino, and T. Tornese (2023): “Blended Identification in Structural VARs,” Discussion paper, mimeo.

Angelini, G., E. Bacchiocchi, G. Caggiano, and L. Fanelli (2019): "Uncertainty across volatility regimes," *Journal of Applied Econometrics*, 34(3), 437–455.

Arias, J., J. Rubio-Ramírez, and D. Waggoner (2018): "Inference Based on SVARs Identified with Sign and Zero Restrictions: Theory and Applications," *Econometrica*, 86(2), 685–720.

Bacchiocchi, E. (2017): "On the identification of interdependence and contagion of financial crises," *Oxford Bulletin of Economics and Statistics*, 79(6), 1148–1175.

Bacchiocchi, E., and L. Fanelli (2015): "Identification in Structural Vector Autoregressive Models with Structural Changes with an Application to U.S. Monetary Policy," *Oxford Bulletin of Economics and Statistics*, 77, 761–779.

Bacchiocchi, E., and T. Kitagawa (2020a): "Locally- but not globally-identified SVARs," Discussion paper, mimeo.

——— (2020b): "SVARs with breaks: Identification and inference," Discussion paper, mimeo.

——— (2021): "On global identication in Structural Vector Autoregressions," Discussion paper, mimeo.

Baumeister, C., and J. Hamilton (2015): "Sign Restrictions, Structural Vector Autoregressions, and Useful Prior Information," *Econometrica*, 83(5), 1963–1999.

Baumeister, C., and J. D. Hamilton (2019): "Structural interpretation of vector autoregressions with incomplete identification: Revisiting the role of oil supply and demand shocks," *American Economic Review*, 109(5), 1873–1910.

Baumeister, C., and G. Peersman (2013): "The role of time-varying price elasticities in accounting for volatility changes in the crude oil market," *Journal of Applied Econometrics*, 28(7), 1087–1109.

Boivin, J., and M. Giannoni (2006): "Has Monetary Policy Become More Effective?," *Review of Economics and Statistics*, 88(3), 445–462.

Brunnermeier, M., D. Palia, K. A. Sastry, and C. A. Sims (2021): "Feedbacks: Financial Markets and Economic Activity," *American Economic Review*, 111(6), 1845–1879.

Gafarov, B., M. Meier, and J. Montiel-Olea (2018): "Delta-method Inference for a Class of Set-identified SVARs," *Journal of Econometrics*, 203(2), 316–327.

Giacomini, R., and T. Kitagawa (2014): "Inference about Non-Identified SVARs," Discussion paper, CeMMAP working paper 45/14.

——— (2021): "Robust Bayesian Inference for Set-identified Models," *Econometrica*, 89(4), 1519–1556.

Granziera, E., H. Moon, and F. Schorfheide (2018): "Inference for VARs Identified with Sign Restrictions," *Quantitative Economics*, 9(3), 1087–1121.

Hamilton, J., D. Waggoner, and T. Zha (2007): "Normalization in econometrics," *Econometric Reviews*, 26(2-4), 221–252.

Inoue, A., and B. Rossi (2011): "Identifying the Sources of Instabilities in Macroeconomic Fluctuations," *Review of Economics and Statistics*, 93(4), 1186–1204.

Kilian, L. (2009): "Not all oil price shocks are alike: Disentangling demand and supply shocks in the crude oil market," *American Economic Review*, 99(3), 1053–69.

Kilian, L., and H. Lütkepohl (2017): *Structural Vector Autoregressive Analysis*. Cambridge, UK: Cambridge University Press.

Kilian, L., and D. P. Murphy (2012): "Why agnostic sign restrictions are not enough: understanding the dynamics of oil market VAR models," *Journal of the European Economic Association*, 10(5), 1166–1188.

——— (2014): "The role of inventories and speculative trading in the global market for crude oil," *Journal of Applied Econometrics*, 29(3), 454–478.

Klein, R., and F. Vella (2010): "Estimating a Class of Triangular Simultaneous Equations Models without Exclusion Restrictions," *Journal of Econometrics*, 154(2), 154–164.

Lanne, M., and H. Lütkepohl (2008): "Identifying Monetary Policy Shocks via Changes in Volatility," *Journal of Money, Credit and Banking*, 40, 1131–1149.

Lanne, M. L., H. Lütkepohl, and K. Maciejowska (2010): "Structural Vector Autoregressions with Markov Switching," *Journal of Economic Dynamics and Control*, 34, 21–131.

Lewbel, A. (2012): "Using heteroskedasticity to identify and estimate mismeasured and endogenous regressor models," *Journal of Business and Economic Statistics*, 30(1), 67–80.

Lewis, D. J. (2021): "Identifying Shocks via Time-Varying Volatility," *Review of Economic Studies*, forthcoming.

Lütkepohl, H. (2013): "Identifying Structural Vector Autoregressions via Changes in Volatility," in *VAR Models in Macroeconomics – New Developments and Applications: Essays in Honor of Christopher A. Sims*, ed. by T. B. Fomby, L. Kilian, and A. Murphy, vol. 32, chap. 5, pp. 169–203. Emerald Group Publishing Limited.

Lütkepohl, H., M. Meitz, A. Netšunajev, and P. Saikkonen (2020): "Testing identification via heteroskedasticity in structural vector autoregressive models," *The Econometrics Journal*, forthcoming.

Lütkepohl, H., and A. Netšunajev (2014): "Disentangling demand and supply shocks in the crude oil market: How to check sign restrictions in structural VARs," *Journal of Applied Econometrics*, 29(3), 479–496.

Lütkepohl, H., and A. Netšunajev (2017): "Structural vector autoregressions with heteroskedasticy: a review of different volatility models," *Econometrics and Statistics*, 1, 2–18.

Lütkepohl, H., and T. Schlaak (2021): "Heteroscedastic proxy vector autoregressions," *Journal of Business & Economic Statistics*, pp. 1–14.

Magnus, J., and H. Neudecker (2007): *Matrix differential calculus with applications in statistics and econometrics.* John Wiley and Sons, Inc.

Magnusson, L. M., and S. Mavroeidis (2014): "Identification using stability restrictions," *Econometrica*, 82(5).

Muirhead, R. J. (ed.) (1982): *Aspects of Multivariate Statistical Theory.* Hoboken, NJ: John Wiley & Sons.

Rigobon, R. (2003): "Identification through Heteroskedasticity," *Review of Economics and Statistics*, 85(4), 777–792.

Rubio-Ramírez, J. F., D. F. Waggoner, and T. Zha (2010): "Structural Vector Autoregressions: Theory of Identification and Algorithms for Inference," *Review of Economic Studies*, 77, 665–696.

Sentana, E., and G. Fiorentini (2001): "Identification, estimation and testing of conditionally heteroskedastic factor models," *Journal of Econometrics*, 102(2), 143–164.

Sims, C. (2020): "SVAR Identification through Heteroskedasticity with Misspecified Regimes," Unpublished, Princeton University.

Sims, C., and T. Zha (2006): "Were There Regime Switches in U.S. Monetary Policy?," *American Economic Review*, 96(1), 54–81.

Uhlig, H. (2005): "What are the Effects of Monetary Policy? Results from an Agnostic Identification Procedure," *Journal of Monetary Economics*, 52, 381–419.

Waggoner, D., and T. Zha (2003): "Likelihood preserving normalization in multiple equation models," *Journal of Econometrics*, 114(2), 329–347.

Zhou, X. (2020): "Refining the workhorse oil market model," *Journal of Applied Econometrics*, 35(1), 130–140.

A Proofs

We first introduce some notation that will be used in the following proofs. For any reduced-form parameter $\phi \in \Phi$, let λ_i be an eigenvalue of the eigenproblem as in Definition 1 with algebraic multiplicity $g(\lambda_i) = m_i$, with associated eigenspace $Q(\lambda_i)$ as in Eq. (24), containing $q^i_{j^*}$, the column of Q associated with the j^*-th structural shock (shock of interest). We have defined the zero restrictions on the vectors $\left(q^i_1, \ldots, q^i_{m_i}\right) \in Q(\lambda_i)$ in terms of the matrix $F^i_j(\phi)$, with ϕ-a.s. full row rank equal to f^i_j. Let $Q^{\perp}(\lambda_i)$, instead, be the linear space in $\mathbb{R}^n$, of dimension $(n - m_i)$, whose elements are orthogonal to $Q(\lambda_i)$. A basis for this linear space is given by $\left(v^i_1, \ldots, v_{(n-m_i)}\right)$. We define $\mathcal{F}^{i\perp}_j(\phi)$ the linear subspace of $\mathbb{R}^n$ that is orthogonal to the row vectors of $F^i_j(\phi)$ and to $Q^{\perp}(\lambda_i)$. We let $\mathcal{H}_j(\phi)$ be the half-space in $\mathbb{R}^n$ defined by the sign-normalization constraint $\left\{x \in \mathbb{R}^n \,\middle|\, (\sigma^j)\prime x \geq 0\right\}$, with σ^i being the j-th column of $\Omega^{-1}_{1,tr}$. As before, $\mathcal{S}^{n-1}$ indicates the unit sphere in $\mathbb{R}^n$. Finally, given k linearly independent vectors in $\mathbb{R}^n$, $V = \left(v_1, \ldots, v_k\right) \in \mathbb{R}^{n \times k}$, let $\mathcal{P}(V)$ be the linear subspace in $\mathbb{R}^n$, of dimension $(n - k)$ that is orthogonal to the column vectors of V.

Lemma 2 (Diagonalization of symmetric matrices)**.** *Let Ω be a symmetric matrix in $\mathbb{R}^{n \times n}$, then it is diagonalizable, i.e. there exists an orthogonal matrix $Q \in \mathcal{O}(n)$, made of the (unit) eigenvectors of Ω, such that $\Omega Q = QD$, or equivalently, $Q'\Omega Q = D$, where D is diagonal. Moreover, the matrix D contains the (real) eigenvalues of Ω, corresponding to the eigenvectors in Q.*

Proof of Lemma 2. See Magnus and Neudecker (2007), Chapter 1, Theorem 13 (page 17). □

Lemma 3. *In real symmetric matrices the algebraic multiplicity does correspond to the geometric multiplicity.*

Proof of Lemma 3. Let A be and $n \times n$ symmetric matrix whose i-th eigenvalue is represented by λ_i, with algebraic multiplicity equal to $1 < m_i \leq n$. Then, there exists some unit-length eigenvector p_{i1}. Let $B = \begin{pmatrix} p_{i1} & C \end{pmatrix}$ be an orthogonal matrix. Then we have

$$B'AB = \begin{pmatrix} \lambda_i & 0 \\ 0 & C'AC \end{pmatrix}.$$

As the algebraic multiplicity m_i is greater than one, from the characteristic polynomial we have that $\left|C'AC - \lambda_i I_{n-1}\right| = 0$, that implies there will be some non-null vector q such that $\left(C'AC - \lambda_i I_{n-1}\right)q = 0$. Let $p_{i2} = Cq$. It is easy to show that p_{i2} is an eigenvector of A. In fact, from the previous relation $\left(C'AC - \lambda_i I_{n-1}\right)q = 0$ we have $A\,C\,q = \lambda_i C\,q$, that implies $Ap_{i2} = \lambda_i p_{i2}$. Moreover, by construction, p_{i1} will be orthogonal to p_{i2}. It will be possible, thus, to define a new B of the form $B = \begin{pmatrix} p_{i1} & p_{i2} & C \end{pmatrix}$ such that

$$B'AB = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & C'AC \end{pmatrix}.$$

and proceed as before for all the algebraic multiplicity of λ_i. The matrix $E = \left(p_{i1}, \ldots, p_{im_i}\right)$ will be a basis for the eigenspace of A associated with λ_i, and the dimension of such space will be clearly m_i, being all the columns of E orthogonal. □

Proof of Theorem 1. Let (C, Λ) be a solution of the equation system different from (C^*, Λ^*) with non singular C. Then, there exists a $n \times n$ matrix A such that $C^* = CA$ holds. Note that A has to be an orthogonal matrix $AA' = I$ as otherwise $\Omega_1 = C^*C^{*\prime} = CC'$ violates. In order for (10) to hold for both (C^*, Λ^*) and (C, Λ),

$$\Omega_2 = C^*\Lambda^*C^{*\prime} = CA\Lambda^*A'C' \tag{47}$$

must hold and, hence, $\Lambda = A\Lambda^*A'$ holds. We therefore investigate the conditions on orthogonal matrix A such that $A\Lambda^*A'$ yields a diagonal matrix with non negative entries.

Let $(\lambda_1^*, \ldots, \lambda_n^*)$ be the diagonal elements of Λ^* and a_{ij} be (i,j)-element of A. Note that non-singularity of Ω_2 implies $\lambda_k^* > 0$ for all $k = 1, \ldots, n$. Noting that the (i,j)-element of $A\Lambda^*A'$ can be expressed as $\sum_{k=1}^{n} \lambda_k^* a_{ik}a_{jk}$, A has to satisfy

$$\begin{cases} \sum_{k=1}^{n} \lambda_k^* a_{ik}a_{jk} = 0, & \forall i \neq j \\ \sum_{k=1}^{n} \lambda_k^* a_{ik}a_{jk} \geq 0, & \forall i = j. \end{cases}$$

The second set of conditions does not at all constrain A, while the first set of conditions constrains A to those such that every row vector in A has only one nonzero element and none of the row vectors in A shares the column-index for the non-zero entry. Combined with orthogonality of A, feasible A's can be therefore represented by $A = PS$. □

Proof of Theorem 4. The proof of the theorem is trivial and completely based on the proof of the Single Value Decomposition for square matrices, see among many others Magnus and Neudecker (2007), pages 19-20. The first point to remark is that, if we call $\Omega_{tr} = \Omega_{1,tr}^{-1}\Omega_{2,tr}$, then $\Lambda^{1/2}$ contains the positive square root of the eigenvalues of $\Omega = \Omega_{1,tr}^{-1}\Omega_{2,tr}\Omega_{2,tr}'\Omega_{1,tr}^{-1\prime} = \Omega_{1,tr}^{-1}\Omega_2\Omega_{1,tr}^{-1\prime}$, as described in Theorem 3. However, for symmetric and non-singular real matrices like Ω, the number of identical eigenvalues (real and different from zero) corresponds to the number of degenerate singular values in Ω_{tr}. As a consequence, if all the elements in $\Lambda^{1/2}$ are distinct, then all the singular values are non-degenerate, and the singular value decomposition is unique (Q and Q_2 are unique), up to multiplication of a specific column of Q and Q_2 by -1, or changing the ordering of the elements in $\Lambda^{1/2}$ (or Λ). □

Proof of Theorem 8. The proof of the theorem takes inspiration from Rubio-Ramírez, Waggoner, and Zha (2010) (proof of their Theorem 7). When the two covariance matrices are perfectly proportional, or even equal, then the condition in the theorem collapses to the well known condition in Rubio-Ramírez, Waggoner, and Zha (2010) and Bacchiocchi and Kitagawa (2021), and the proof is thus immediate. On the other side, if all eigenvalues are distinct, i.e. $k = n$, then the results of Theorem 3 apply. Similar results apply for all eigenvalues with algebraic multiplicity equal to one, i.e. $m_i = 1$. According to Lemma 3, the geometric multiplicity is equal to the algebraic one, and thus, if $m_i = 1$ the eigenspace associated to such eigenvalues will generate spaces of dimension one, each. Imposing unit length and sign normalization allows to uniquely identify such vectors. Moreover, given

Lemma 2, such vectors will be mutually orthogonal. They will constitute the columns of Q associated with the eigenvalues of multiplicity one.

We will now turn to the case when λ has multiplicity greater than 1. Let λ_i be characterized by algebraic multiplicity $m_i \leq 2$. Given Lemma 3, the m_i associated eigenvectors, although not unique, represent an orthonormal basis for the subspace $Q(\lambda_i)$, of dimension m_i in $\mathbb{R}^n$. For the condition in Theorem 8 to be sufficient, we need to prove that imposing such particular pattern of zero restrictions allows to uniquely pin down orthonormal vectors lying in $Q(\lambda_i)$. Let $V(\lambda_i) = \left(v_1^i, \ldots, v_{m_i}^i\right)$ be a basis for the eigenspace associated to λ_i. The identified vectors $\left(q_1^i, \ldots, q_{m_i}^i\right)$ must satisfy the following conditions:

- they must be a linear combination of the orthonormal basis identified through the eigen-problem;
- they must be orthogonal each other;
- they must satisfy the zero and normalization restrictions;
- they must have unit length.

We can think of writing a system of equations. The number of unknowns is m_i for each vector q_j^i, $j = 1, \ldots, m_i$. Let the ordering of the vectors be fixed according to the number of restrictions, from the more the less constrained. We start from the first and most constrained vector

$$q_1^i = v_1^i x_1 + \cdots + v_{m_i}^i x_{m_i}$$

that is subjected to the $m_i - 1$ zero restrictions $F_1^i(\phi) q_1^i = 0$. Substituting the definition of q_i according to the previous relation, we simply obtain:

$$\begin{bmatrix} F_1^i(\phi) v_1^i & \cdots & F_1^i(\phi) v_{m_i}^i \end{bmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{m_i} \end{pmatrix} = 0. \tag{48}$$

As $\operatorname{rank}\left(F_1^i(\phi)\right) = m_i - 1$, ϕ-almost surely (a.s.), then the matrix $\begin{bmatrix} F_1^i(\phi) v_1^i & \cdots & F_1^i(\phi) v_{m_i}^i \end{bmatrix}$ projecting m_i orthogonal vectors in $\mathbb{R}^n$ onto an $m_i - 1$ dimensional space, will generate $m_i - 1$ linearly independent vectors in $\mathbb{R}^{m_i}$. As a consequence, there will be just a unidimensional space in $\mathbb{R}^{m_i}$ that is orthogonal to $F_1^i(\phi)$. Let $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_{m_i})'$ be a unit vector representing a basis for this vector space, then

$$q_i = v_1^i \alpha \tilde{x}_1 + \cdots + v_{m_i}^i \alpha \tilde{x}_{m_i}.$$

However, q_i must have unit length, thus

$$\begin{aligned} q_1' q_1 = 1 \implies\ & \alpha^2 \left(v_1^i \tilde{x}_1 + \cdots + v_{m_i}^i \tilde{x}_{m_i}\right)' \left(v_1^i \tilde{x}_1 + \cdots + v_{m_i}^i \tilde{x}_{m_i}\right) = 1 \\ \implies\ & \alpha^2 \left(\tilde{x}_1^2 + \cdots + \tilde{x}_{m_i}^2\right) = 1 \\ \implies\ & \alpha^2 = 1 \implies \alpha = \pm 1, \end{aligned}$$

indicating that there will be two opposite vectors candidates for q_1^i, one of which, however, is ruled out by the normality sign restrictions. The following step consists in determining

$q_2^i \in Q(\lambda_i)$, orthogonal to q_1^i and satisfying the $(m_i - 2)$ restrictions $F_2^i(\phi) q_2^i = 0$. We can think of a system of equations of the form

$$\begin{cases} F_2^i(\phi) q_2^i &= 0 \\ q_1^{i\prime} q_2^i &= 0 \end{cases}$$

where q_1^i is known from the previous step. Substituting for the definition of q_2^i in terms of the basis of the vector space it belongs, i.e. $V(\lambda_i) = \left(v_1^i, \ldots, v_{m_i}^i\right)$, with simple algebra, the system can also be written as

$$\left[\begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix} v_1^i \quad \cdots + \quad \begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix} v_{m_i}^i \right] \begin{pmatrix} x_1 \\ \vdots \\ x_{m_i} \end{pmatrix} = 0.$$

According to the assumed non-redundancy of the restrictions, as in Definition 3, the quantity $\begin{pmatrix} F_2^i(\phi) \\ q_1^i \end{pmatrix}$ has full row rank $m_i - 1$, we are exactly in the same situation as in Eq. (48). We can thus proceed as before and obtain two potential opposite unit-length vectors q_2^i, one of which, however, is ruled out by the sign normalization restrictions. This strategy allows to prove the point identification of all the $\left(q_1^i, \ldots, q_{m_i}^i\right)$ vectors associated with the i-th multiple eigenvalue λ_i. The *sufficient* direction of the condition is thus proved.

The *necessary* part of the condition can be proved as follows. Let the parameter $(A_0, A_+, \Lambda) \in \mathcal{A}^r(\phi)$ be point identified. As a consequence, the set of admissible orthogonal matrices $\mathcal{Q}(\phi\,|F, S)$ will be a singleton, say Q. If the i-th column of Q is the eigenvector associated to an eigenvalue with no multiplicity ($m_i = 1$), then it is unique and no zero restriction is needed, thus $f_j^i = m_i - j = 1 - 1 = 0$ as predicted by the theorem. For those columns of Q associated with an eigenvalue with multiplicity $m_i > 1$ the condition can be directly proved by using Lemma 4 in Bacchiocchi and Kitagawa (2021), that extends Lemma 9 in Rubio-Ramírez, Waggoner, and Zha (2010) to the case of non-redundant restrictions. □

Proof of Theorem 9. Let j^* the shock of interest, that is associated with $q_{j^*}^i \in Q(\lambda_i)$, the eigenspace related to λ_i, with multiplicity m_i. According to Definition 1, the space $Q(\lambda_i)$ is orthogonal to the linear space generated by all the other eigenvectors of the eigenproblem, that we denote by $Q^{\perp}(\lambda_i)$. The dimension of $Q^{\perp}(\lambda_i)$ is $(n - m_i)$ and let the vectors $(v_1^i, \ldots, v_{(n-m_i)}^i)$ be a possible basis. Moreover, let the vectors $(q_1^i, \ldots, q_{m_i}) \in Q(\lambda_i)$ be defined in the following recursive way:

$$\begin{array}{lll} q_1^i & \in & V_1^i(\phi) \equiv \mathcal{F}_1^{i\perp}(\phi) \cap \mathcal{H}_1(\phi) \cap \mathcal{S}^{n-1} \\ q_2^i & \in & V_2^i(\phi, q_1^i) \equiv \mathcal{F}_2^{i\perp}(\phi) \cap \mathcal{H}_2(\phi) \cap \mathcal{P}(q_1^i) \cap \mathcal{S}^{n-1} \\ q_3^i & \in & V_3^i(\phi, Q_{1:2}^i) \equiv \mathcal{F}_3^{i\perp}(\phi) \cap \mathcal{H}_3(\phi) \cap \mathcal{P}(Q_{1:2}^i) \cap \mathcal{S}^{n-1} \\ \vdots & & \\ q_{m_i}^i & \in & V_{m_i}^i(\phi, Q_{1:m_i-1}^i) \equiv \mathcal{F}_{m_i}^{i\perp}(\phi) \cap \mathcal{H}_{m_i}(\phi) \cap \mathcal{P}(Q_{1:m_i-1}^i) \cap \mathcal{S}^{n-1} \end{array} \tag{49}$$

where the generic $\mathcal{F}_j^{i\perp}(\phi)$ is the linear subspace of $\mathbb{R}^n$ that is orthogonal to the row vectors of

$F_j^i(\phi)$ and to $Q^{\perp}(\lambda_i)$. The dimension of $\mathcal{F}_j^{i\perp}(\phi)$ is $\dim\big(\mathcal{F}_j^{i\perp}(\phi)\big) = n-(n-m_i)-f_j^i = m_i - f_j^i$.

If $j^* = 1$, then we know that $f_1^i \leq m_i - 1$, and $\mathcal{F}_1^{i\perp}(\phi) \cap \mathcal{H}_1(\phi)$ is the half-space of the linear subspace of $\mathbb{R}^2$ with dimension $m_i - f_1^i \geq 1$. As a consequence, $V_1^i(\phi, q_i^i)$ is non empty for every $\phi \in \varPhi$. Similarly, if $j^* = 2, \ldots, m_i$, $\mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:j^*})$ is the half-space of the linear subspace of $\mathbb{R}^n$ of dimension at least $m_i - f_{j^*}^i - (j^* - 1) \geq 1$, being $f_{j^*}^i \leq m_i - j^*$. For $j^* = 1, \ldots, m_i$, thus, $V_{j^*}^i(\phi, Q_{1:j^*-1}^i)$ is non empty and, as a consequence, $Q(\phi, F)$ will be non empty, too. Non emptiness of the impulse responses is a direct consequence. Concerning the boundedness, it immediately follows from the fact that $|r_{lj^*}^h| \leq \|c_{lh}(\phi)\| \leq \infty$ for any $l \in \{1, \ldots, n\}$, $j^* \in \{1, \ldots, m_i\}$ and $h = 0, 1, 2, \ldots$, where $\|c_{lh}(\phi)\| \leq \infty$ is guaranteed by the invertibility of the VAR characteristic polynomial. The first part of the proof is thus complete. We can move to prove the convexity of the identified set.

Let $j^* = 1$ and $f_1^i \leq m_i - 1$ (condition 1). Being $V_1^i(\phi)$ the intersection of an half-space of dimension at least 2 and an unit sphere it is path connected for all values of the reduced-form parameters ϕ. Hence, the identified set $r_{l1}^h = c_{lh}(\phi) q_1^i$ will be an interval, being the impulse response a continuous function with a path connected domain always an interval. Concerning condition (2) of the Theorem, we can prove the result by applying Lemma A.1 in Giacomini and Kitagawa (2014). According to the definition of $\mathcal{F}_j^{i\perp}(\phi)$, that collects not simply vectors orthogonal to the row vectors of $F_j^i(\phi)$, but also orthogonal to all other vectors belonging to $Q^{\perp}(\lambda_i)$, Lemma A.1 in Giacomini and Kitagawa (2014) allows to simplify the set of admissible $q_{j^*}^i$. In fact, if we define $\mathcal{E}_{j^*}^i(\phi)$ the set of admissible $q_{j^*}^i$, then using Lemma A.1 we derive that $\mathcal{E}_{j^*}^i(\phi) = \mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{S}^{n-1}$. Hence, being $\mathcal{E}_{j^*}^i(\phi)$ the intersection of a half-space of a linear subspace with dimension $m_i - f_{j^*}^i \geq j^* \geq 2$ with the unit sphere, it is a path connected set on $\mathcal{S}^{n-1}$, and the convexity of the identified set immediately follows.

In a similar way we can prove the result for condition (3) of the theorem. In this respect we can use Lemma A.2 in Giacomini and Kitagawa (2014), that, based on our definition of $\mathcal{F}_j^{i\perp}(\phi)$, allows to derive the set of potential $q_{j^*}^i$ subject to condition (3), i.e. $\mathcal{E}_{j^*}^i(\phi) = \mathcal{F}_{j^*}^{i\perp}(\phi) \cap \mathcal{H}_{j^*}(\phi) \cap \mathcal{P}(Q_{1:k}^i) \cap \mathcal{S}^{n-1}$. According to this definition, $\mathcal{E}_{j^*}^i(\phi)$ is the intersection of a half-space of a linear subspace of dimension $n-(n-m_i)-f_{j^*}^i-k > j^*-k \geq 2$ and a unit sphere, and, thus, it is a path connected set on $\mathcal{S}^{n-1}$. The convexity of the identified set, hence, clearly holds.

In all cases, the convexity of the identified set depends on $\phi \in \varPhi$, being the multiplicity of λ_i equal to m_i only ϕ-a.s. Thus, convexity of the identified set holds ϕ-a.s. $\square$

Proof of Theorem 10. The proof builds on Lemma A.2 in Giacomini and Kitagawa (2014). Let first $j^* = 1$ and $f_1^i < m_i - 1$. According to the notation introduced in Eq. (49), the set of admissible q_1^i becomes $V_1^i(\phi) \cap \big\{x \in \mathbb{R}^n : S_1^i(\phi)\, x \geq 0\big\}$. Moreover, let $\tilde{q}_1^i 1$ be another arbitrary unit length vector satisfying the zero, sign normalization and sign restrictions. Clearly, according to the sign restrictions, it must hold that $q_1^i \neq \tilde{q}_1^i 1$. The intuition for proving the result consists in observing that any weighted average of the two admissible vectors, with positive weights summing to one, continues to belong to the set. Then, if we define

$$q_i^i(\delta) = \frac{\delta q_1^i + (1-\delta)\tilde{q}_1^i 1}{\big\|\delta q_1^i + (1-\delta)\tilde{q}_1^i 1\big\|}, \qquad \delta \in [0\,,\,1] \tag{50}$$

it represents a connected path in $V_1^i(\phi) \cap \left\{x \in \mathbb{R}^n : S_1^i(\phi)\, x \geq 0\right\}$, as the denominator is always different then zero, given that $q_1^i \neq \tilde{q}_1^i 1$. Any generic couple of admissible vectors, thus, can be connected by a connected path. The convexity of the impulse response, thus, immediately follows. We now assume that condition (2) in Theorem 9 holds. Now, let $\mathcal{E}_{j^*}^i(\phi)$ be the set of admissible $q_{j^*}^i$ satisfying zero, sign normalization and sign restrictions. Let $q_{j^*}^i$ and $\tilde{q}_{j^*}^i$ be two arbitrary vectors belonging to $\mathcal{E}_{j^*}^i(\phi)$. Clearly, due to the sign restrictions, $q_{j^*}^i \neq \tilde{q}_{j^*}^i$. As before, we consider a path between these two vectors as follows

$$q_{j^*}^i(\delta) = \frac{\delta q_{j^*}^i + (1-\delta)\tilde{q}_{j^*}^i}{\left\|\delta q_{j^*}^i + (1-\delta)\tilde{q}_{j^*}^i\right\|}, \qquad \delta \in [0\,,\,1] \tag{51}$$

which is a continuous path on the unit sphere as the denominator is always different than zero, being $q_{j^*}^i \neq \tilde{q}_{j^*}^i$. Now, the path connectedness of $\mathcal{E}_{j^*}^i(\phi)$ depends on whether it is possible to obtain an admissible set of vectors $Q^i(\delta) = \left(q_1^i(\delta), \ldots, q_{m_i}^i(\delta)\right)$ whose j^*-th element is represented by the $q_{j^*}^i(\delta)$ vector. Conditional on a basis $\left(v_1^i, \ldots, v_{(n-m_i)}^i\right)$ for the space $Q^{\perp}(\lambda_i)$, the first k vectors in $Q^i(\delta)$, $k = 1, \ldots, j^*-1$ can be obtained through the solutions of the recursive system of equations

$$\begin{pmatrix} F_s^i(\phi) \\ v_1^i \\ \vdots \\ v_{n-m_i} \\ q_1^i(\delta) \\ \vdots \\ q_{k-1}^i(\delta) \\ q_{j^*}^i(\delta) \end{pmatrix} q_k^i(\delta) = 0, \qquad \delta \in [0\,,\,1] \tag{52}$$

satisfying the further sign normalization restriction. As the rank of the matrix in the system is at most $n - m_i + k + f_k^i$, that is always less then n because $f_k^i < m_i - k$, a solution always exists. The remaining vectors for $j^* + 1, \ldots, m_i$ can be obtained recursively by extending the system in Eq. (52). The set $\mathcal{E}_{j^*}^i(\phi)$, thus, is non empty and path connected. The convexity of the impulse response identified set immediately follows.

Concerning point (2) of the theorem, let the zero restriction satisfy condition (3) of Theorem 9, and let $\left(q_1^i, \ldots, q_k^i\right)$ be the exactly identified vectors, common to all admissible $Q(\lambda)$ matrices. As before, we chose two arbitrary vectors $q_{j^*}^i$ and $\tilde{q}_{j^*}^i$, both satisfying the zero, sign normalization and sign restrictions, and obtain the further vector $\tilde{q}_{j^*}^i(\delta)$ as in Eq. (51). We can thus construct the set $Q^i(\delta)$, whose first k columns are given by $\left(q_1^i, \ldots, q_k^i\right)$. Conditional on the choice of δ, for $s = k+1, \ldots, j^*-1$, we can recursively derive $q_s^i(\delta)$ by solving

$$\begin{pmatrix} F_s^i(\phi) \\ v_1^i \\ \vdots \\ v_{n-m_i} \\ q_1^i \\ \vdots \\ q_k^i \\ q_{k+1}^i(\delta) \\ \vdots \\ q_{s-1}^i(\delta) \\ q_{j^*}^i(\delta) \end{pmatrix} q_s^i(\delta) = 0, \qquad \delta \in [0\,,\,1] \tag{53}$$

where $q_s^i(\delta)$ satisfies the sign normalization restriction, and where $\big(v_1^i, \ldots, v_{(n-m_i)}^i\big)$ is a basis for the space $Q^{\perp}(\lambda_i)$. The system always admits a solution, being the rank of the matrix less than n by the assumption on the number of zero restrictions on $q_{k+1}, \ldots, q_{j^*-1}^i$, being $f_s^i < m_i - s$, for $s = 1, \ldots, j^* - 1$. The remaining $q_{j^*+1}^i(\delta), \ldots, q_{m_i}^i(\delta)$ vectors can be recursively derived by extending the system in Eq. (53). Once proved on how to derive $Q^i(\delta)$ as a function of $\delta \in [0\,,\,1]$, the set $\mathcal{E}_{j^*}^i(\phi)$ is path connected, and the associated impulse response identified set is convex for every variable at any horizon. $\square$

Proof of Lemma 1. The (squared of the) Frobenius norm states that

$$\left\|\Omega - \tilde{\Omega}\right\|_F^2 = \sum_{i,j}^{n} \left(\Omega_{ij} - \tilde{\Omega}_{ij}\right)^2 = \mathrm{tr}\left[(\Omega - \tilde{\Omega})'(\Omega - \tilde{\Omega})\right].$$

However, given the definition of Ω and $\tilde{\Omega}$

$$\Omega = Q\Lambda Q' \quad \text{ and } \quad \tilde{\Omega} = Q\tilde{\Lambda}Q',$$

we have that

$$\begin{aligned} \left\|\Omega - \tilde{\Omega}\right\|_F^2 &= \mathrm{tr}\left[(Q\Lambda Q' - Q\tilde{\Lambda}Q')'(Q\Lambda Q' - Q\tilde{\Lambda}Q')\right] \\ &= \mathrm{tr}\left[(Q(\Lambda - \tilde{\Lambda})Q')'(Q(\Lambda - \tilde{\Lambda})Q')\right] \\ &= \mathrm{tr}\left[Q(\Lambda - \tilde{\Lambda})(\Lambda - \tilde{\Lambda})Q'\right] \\ &= \mathrm{tr}\left[QQ'(\Lambda - \tilde{\Lambda})^2\right] \\ &= \mathrm{tr}\left[(\Lambda - \tilde{\Lambda})^2\right] \\ &= \sum_{h=1}^{m} (\lambda_h - \tilde{\lambda})^2. \end{aligned}$$

Clearly, this is minimized when $\tilde{\lambda} = \frac{1}{m}\sum_{h=1}^{m} \lambda_h$, i.e. the mean of the eigenvalues corresponding to those restricted to be equal. $\square$

B Proofs of Theorems on bivariate SVARs and HSVARs

Proof of Theorem 5. Given the decomposition of A_0 as in Eq. (16), then

$$\beta = -[A_0]_{(1,2)} \,/\, [A_0]_{(1,1)} = -\left(q_1'\omega_2\right) / \left(q_1'\omega_1\right) \tag{54}$$

$$\alpha = -[A_0]_{(2,1)} \,/\, [A_0]_{(2,2)} = -\left(q_2'\omega_1\right) / \left(q_2'\omega_2\right) \tag{55}$$

where ω_1 and ω_2 are defined as in Eq. (17) while q_1 and q_2 are the two columns of the orthogonal matrix Q. The proof of the theorem is extremely intuitive when observing the two graphs in Figure 1 and Figure 2. Consider the situation of $\omega_{pq} \geq 0$ (Case I, Figure 1), first. In both panels we report the observable ω_1 and ω_2 vectors, compatible with $\omega_{pq} \geq 0$. In the left panel we focus on all the admissible β. First of all, if we look at the definition of β in Eq. (54), it is very simple to obtain the two values of q_1 featuring the extreme values of $\beta = -\infty$ and $\beta = \infty$. In both cases, q_1 must be orthogonal to ω_1; however, in one case the numerator of β is negative (solid line), while in the other the numerator is positive (dotted line). The vector q_1 featuring $\beta = 0$ has to be orthogonal to ω_2. This generates two potential q_1 vectors, i.e. $q_1 = (0, 1)$ and $q_1 = (0, -1)$. The latter, however, has to be discarded given Assumption 1. It can be deduced, thus, that the admissible β are those generated by the q_1 vectors lying on the right half of the unit circle (light red area) and, without any restriction, $\beta \in (-\infty\,, \infty)$. The arc in red, instead, highlight the feasible q_1 vectors consistent with the sign restriction $\beta \leq 0$, as reported in Assumption 2. In the right panel, instead, we focus on the coefficient α. Even in this case we report the two observable vectors ω_1 and ω_2. Following the same strategy as before, we determine all the admissible vectors q_2 compatible with the α coefficient. However, keeping in mind that q_1 and q_2 must be orthogonal by construction, it can be remarked that, when q_1 reaches the two extreme values $\beta = -\infty$ and $\beta = \infty$, it can no longer rotate counterclockwise, as it is at odds with Assumption 1. This implies that q_2 can not reach the limit case of $\alpha = \infty$. Without any further restriction, $\alpha \in (-\infty\,, \omega_q^2/\omega_{pq})$, where this upper bound of the interval is obtained by substituting in the definition of α in Eq. (55) the value of q_2 that is orthogonal to q_1 featuring $\beta = -\infty$ (or, equivalently, the one featuring $\beta = \infty$), i.e. $q_2 = -\frac{\omega_1}{\|\omega_1\|}$, being $\|\omega_1\|$ the Euclidean norm of ω_1. Thus, with simple algebra, we obtain

$$\alpha = -\left(q_2'\omega_1\right) / \left(q_2'\omega_2\right) = -\left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_1)\right) \Big/ \left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_2)\right) = \frac{\omega_q^2}{\omega_{pq}}. \tag{56}$$

If, instead, we consider the sign restrictions in Assumption 2, then the admissible vectors for q_1 and q_2 are indicated in red and green, respectively. Specifically, being q_2 orthogonal to q_1, it must be in between ω_2 and $-\omega_1$ (green arc), providing thus a 'natural' restriction on the set of possible α's consistent with the two sign restrictions. In particular, the lower bound of the identified set can be obtained through the definition of α in Eq. (55) when q_2 is the unitary vector parallel to ω_2, i.e. $q_2 = (0 \;\; 1)'$. This leads to

$$\alpha = -\left(q_2'\omega_1\right) / \left(q_2'\omega_2\right) = -\Big((0 \;\; 1)'\,\omega_1\Big) \Big/ \Big((0 \;\; 1)'\,\omega_2\Big) = \frac{\omega_{pq}}{\omega_p^2}. \tag{57}$$

The second case, when $\omega_{pq} < 0$, can be addressed in the same way, but now the two sign restrictions in Assumption 2 induce an identified set for β. In particular, the upper bound of the identified set for β can be obtained by using the definition in Eq. (54), when q_1 is a unitary vector parallel to ω_1, while the lower bound can be obtained when q_1 is a unitary vector parallel to ω_2. Simple algebra provides the result for Case II in the theorem. $\square$

Proof of Corollary 1. We know that when $\omega_{pq} \geq 0$ and $\beta = 0$, from Eq. (54), $q_1' \, \omega_2 = 0$. As a consequence, from Figure 1 (left panel), this implies $q_2 = (0 \;\; 1)'$. Thus, substituting for q_2 in Eq. (eq:alpha) and using the definition of ω_1 and ω_2 in Eq. (17) leads to the same result as in Eq. (57), proving thus the first part of the corollary.

When $\omega_{pq} < 0$ and $\alpha = 0$, then $q_2' \, \omega_1 = 0$. This implies that q_1 will be the unit vector parallel to ω_1, i.e. $q_1 = \frac{\omega_1}{\|\omega_1\|}$. Substituting in the definition of β in Eq. (54), leads to

$$\beta = -\left(q_1'\omega_2\right) / \left(q_1'\omega_1\right) = -\left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_2)\right) \Big/ \left(\frac{1}{\|\omega_1\|}(\omega_1'\omega_1)\right) = \frac{\omega_{pq}}{\omega_q^2}. \tag{58}$$

which proves the second part of the corollary. $\square$

Proof of Theorem 7. Based on the definition of Ω_1 and Ω_2, the first step is to calculate the analytical expression for $\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1\prime}$, i.e.

$$\begin{aligned}
\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1\prime} &= \begin{pmatrix} \frac{1}{\omega_{p,1}} & 0 \\ -\frac{\omega_{pq,1}}{\omega_{p,1}^2}\gamma_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \omega_{p,2}^2 & \omega_{pq,2} \\ \omega_{pq,2} & \omega_{p,2}^2 \end{pmatrix} \begin{pmatrix} \frac{1}{\omega_{p,1}} & -\frac{\omega_{pq,1}}{\omega_{p,1}^2}\gamma_1 \\ 0 & \gamma_1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\omega_{p,2}^2}{\omega_{p,1}} & \frac{-\omega_{p,2}^2\omega_{pq,1}+\omega_{pq,2}\omega_{p,1}^2}{\omega_{p,1}^3}\gamma_1 \\ \frac{-\omega_{p,2}^2\omega_{pq,1}+\omega_{pq,2}\omega_{p,1}^2}{\omega_{p,1}^3}\gamma_1 & \frac{\omega_{pq,1}^2\omega_{p,2}^2-2\omega_{pq,1}\omega_{pq,2}\omega_{p,1}^2+\omega_{p,1}^4\omega_{p,2}^2}{\omega_{p,1}^4}\gamma_1^2 \end{pmatrix}
\end{aligned}$$

with γ_1 defined as in Eq. (21). The following step is to calculate the eigenvalues of the previous matrix, that, after some algebra, corresponds to find the solutions of the following quadratic equation of the standard form $a\lambda^2 + b\lambda + c = 0$:

$$\lambda^2 + \left(\frac{-\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2 + 2\omega_{pq,1}\omega_{pq,2}}{\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2}\right)\lambda + \left(\frac{\omega_{p,2}^2\omega_{q,2}^2 - \omega_{pq,2}^2}{\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2}\right) = 0. \tag{59}$$

In solving the quadratic equation it is crucial to focus on the discriminant $\Delta = b^2 - 4ac$ of the equation

$$\Delta = \frac{\left(\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2\right)^2 + 4\left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right)\left(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}\right)}{\left(\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2\right)^2}. \tag{60}$$

Given that the original matrix $\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1\prime}$ is symmetric, then the two eigenvalues are clearly real, and this implies that the discriminant will be not negative. However, if we want the solutions to be distinct (distinct eigenvalues), then we need to find the conditions for Δ to be strictly positive. Firstly, the denominator in Eq. (60) is clearly a real positive number beign the square of the determinant of Ω_1, that is clearly different from zero. The condition of distinct eigenvalues, thus, has to be find on the positiveness of the numerator of Eq. (60). The first term of the sum is clearly non-negative, and, if we show that the second one cannot be negative, too, then the eigenvalues will be clearly distinct as $\Delta > 0$. The non-negativeness of the second term of Eq. (60), with simple algebra, can be seen as:

$$\left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right)\left(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}\right) \geq 0. \tag{61}$$

It immediately emerges that if $\omega_{pq,1}$ and $\omega_{pq,2}$ are of different sign, the previous quantity becomes negative. If, instead, they maintain the same sign, we need to consider the two terms separately

$$\omega_{p,1}^2 \omega_{pq,2} - \omega_{p,2}^2 \omega_{pq,1} \geq 0 \iff \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{p,2}^2}{\omega_{p,1}^2} \tag{62}$$

$$\omega_{q,1}^2 \omega_{pq,2} - \omega_{q,2}^2 \omega_{pq,1} \geq 0 \iff \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{q,2}^2}{\omega_{q,1}^2}. \tag{63}$$

In order to prove when these quantities are positive, it can be useful to consider the definition of ω_1 and ω_2 as a function of the structural parameters contained in $C = A_0^{-1}$, i.e.

$$\begin{aligned} \varOmega_1 &= CC' = \begin{pmatrix} c_{11}^2 + c_{12}^2 & c_{11}c_{21} + c_{12}c_{22} \\ c_{11}c_{21} + c_{12}c_{22} & c_{21}^2 + c_{22}^2 \end{pmatrix} && (64) \\ \varOmega_2 &= C\varLambda C' = \begin{pmatrix} c_{11}^2 \varLambda_{11} + c_{12}^2 \lambda_{22} & c_{11}c_{21}\varLambda_{11} + c_{12}c_{22}\varLambda_{22} \\ c_{11}c_{21}\varLambda_{11} + c_{12}c_{22}\varLambda_{22} & c_{21}^2\varLambda_{11} + c_{22}^2\varLambda_{22} \end{pmatrix} && (65) \end{aligned}$$

where

$$\varLambda = \begin{pmatrix} \lambda_{11} & 0 \\ 0 & \lambda_{22} \end{pmatrix} = \begin{pmatrix} \sigma_{\varepsilon_2}^2/\sigma_{\varepsilon_1}^2 & 0 \\ 0 & \sigma_{\eta_2}^2/\sigma_{\eta_1}^2 \end{pmatrix}$$

and collects the relative shifts in the variances of the structural shocks across the two regimes. From these relationships we obtain

$$\begin{aligned} \frac{\omega_{pq,2}}{\omega_{pq,1}} &= \frac{c_{11}c_{21}\varLambda_{11} + c_{12}c_{22}\varLambda_{22}}{c_{11}c_{21} + c_{12}c_{22}} \\ \frac{\omega_{p,2}^2}{\omega_{p,1}^2} &= \frac{c_{11}^2\varLambda_{11} + c_{12}^2\lambda_{22}}{c_{11}^2 + c_{12}^2} \\ \frac{\omega_{q,2}^2}{\omega_{q,1}^2} &= \frac{c_{21}^2\varLambda_{11} + c_{22}^2\varLambda_{22}}{c_{21}^2 + c_{22}^2} \end{aligned}$$

that allow to investigate the previous inequalities in Eq.s (62)-(63) as follows

$$\begin{aligned} \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{p,2}^2}{\omega_{p,1}^2} &\iff \frac{c_{11}c_{21}\varLambda_{11} + c_{12}c_{22}\varLambda_{22}}{c_{11}c_{21} + c_{12}c_{22}} \geq \frac{c_{11}^2\varLambda_{11} + c_{12}^2\lambda_{22}}{c_{11}^2 + c_{12}^2} \\ &\iff c_{12}c_{11}\left(c_{12}c_{21} - c_{11}c_{22}\right)\left(\lambda_{11} - \lambda_{22}\right) \geq 0 && (66) \\ \frac{\omega_{pq,2}}{\omega_{pq,1}} \geq \frac{\omega_{q,2}^2}{\omega_{q,1}^2} &\iff \frac{c_{11}c_{21}\varLambda_{11} + c_{12}c_{22}\varLambda_{22}}{c_{11}c_{21} + c_{12}c_{22}} \geq \frac{c_{21}^2\varLambda_{11} + c_{22}^2\varLambda_{22}}{c_{21}^2 + c_{22}^2} \\ &\iff c_{21}c_{22}\left(c_{11}c_{22} - c_{12}c_{21}\right)\left(\lambda_{11} - \lambda_{22}\right) \geq 0. && (67) \end{aligned}$$

Given that c_{11} and c_{22} are positive by construction, the system of inequalities becomes

$$c_{12}\left(c_{12}c_{21} - c_{11}c_{22}\right)\left(\lambda_{11} - \lambda_{22}\right) \geq 0 \tag{68}$$

$$c_{21}\left(c_{11}c_{22} - c_{12}c_{21}\right)\left(\lambda_{11} - \lambda_{22}\right) \geq 0. \tag{69}$$

At this point it is important to remember that, from the definition of $C = A_0^{-1}$, $c_{12} \geq 0$ and $c_{21} \leq 0$, due to the sign restrictions on α and β. The previous inequalities, thus, are either always jointly satisfied or jointly never, depending on the sign of $(\lambda_{11} - \lambda_{22})$. This result shows that the inequality in Eq. (61) is always satisfied and thus, being the two addends of the discriminant in Eq. (60) both non-negative, the only possibility we have to exclude for having distinct eigenvalues is when both of them are null, i.e. we have the following system of equations:

$$\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2 = 0 \qquad (70)$$

$$\left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right)\left(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}\right) = 0. \qquad (71)$$

for which the solutions are:

$$\frac{\omega_{p,1}^2}{\omega_{p,2}^2} = \frac{\omega_{q,1}^2}{\omega_{q,2}^2}, \quad \frac{\omega_{p,1}^2}{\omega_{p,2}^2} = \frac{\omega_{pq,1}}{\omega_{pq,2}}, \quad \frac{\omega_{pq,1}}{\omega_{pq,2}} = \frac{\omega_{q,1}^2}{\omega_{q,2}^2} \qquad (72)$$

that corresponds to the case $\Omega_1 = a\Omega_2$, that has been excluded in the theorem.

The eigenvectors, instead, can be calculated from the two systems

$$\left(\Omega_{i,tr}^{-1}\Omega_2\Omega_{i,tr}^{-1\prime} - I_2\lambda_i\right) q_i = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad i = \{1,2\} \qquad (73)$$

where λ_i is the i-th eigenvalue and q_i is the i-th eigenvector. Tedious algebra, not reported here to save space, but available from the authors upon request, proves the following result:

$$q_1 = \begin{pmatrix} \frac{\Delta\omega_{p,1}^2 + D_1}{2D_2\left[\left(\Delta\omega_{p,1}^2 + D_1\right)^2/\left(2D_3^2\Delta^2\right)+1\right]^{1/2} D_3} \\ \\ \frac{1}{\left[\left(\Delta\omega_{p,1}^2 + D_1\right)^2/\left(2D_3^2\Delta^2\right)+1\right]^{1/2}} \end{pmatrix}, \quad q_2 = \begin{pmatrix} \frac{-\Delta\omega_{p,1}^2 + D_1}{2D_2\left[\left(\Delta\omega_{p,1}^2 - D_1\right)^2/\left(2D_3^2\Delta^2\right)+1\right]^{1/2} D_3} \\ \\ \frac{1}{\left[\left(\Delta\omega_{p,1}^2 - D_1\right)^2/\left(2D_3^2\Delta^2\right)+1\right]^{1/2}} \end{pmatrix}. \qquad (74)$$

where

$$\begin{aligned}
\Delta &= \left[\left(\omega_{p,1}^2\omega_{q,2}^2 - \omega_{p,2}^2\omega_{q,1}^2\right)^2 + 4\left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right)\left(\omega_{q,1}^2\omega_{pq,2} - \omega_{q,2}^2\omega_{pq,1}\right)\right]^{1/2} \\
D_1 &= -2\omega_{p,2}^2\omega_{pq,1} - \omega_{p,1}^4\omega_{q,2}^2 + 2\omega_{p,1}^2\omega_{pq,1}\omega_{pq,2} + \omega_{p,1}^2\omega_{p,2}^2\omega_{q,1}^2 \\
D_2 &= \left(\omega_{p,1}^2\omega_{q,1}^2 - \omega_{pq,1}^2\right)^{1/2} \\
D_3 &= \left(\omega_{p,1}^2\omega_{pq,2} - \omega_{p,2}^2\omega_{pq,1}\right).
\end{aligned}$$

The q_1 and q_2 unit vectors form the columns of the orthogonal matrix $Q = (q_1\,,\,q_2)$ such that $C = \Omega_{1,tr}Q$.

If all the λs are equal, the relation $Q'\Omega_{1tr}^{-1}\omega_{2tr}\Omega_{2tr}'\Omega_{1tr}^{-1\prime}Q = \Lambda$ will become

$$\begin{aligned}
\Omega_{1tr}^{-1}\Omega_{2tr}\Omega_{2tr}'\Omega_{1tr}^{-1\prime} &= \lambda QQ' \\
\Rightarrow \quad \Omega_{1tr}^{-1}\Omega_{2tr}\Omega_{2tr}'\Omega_{1tr}^{-1\prime} &= \lambda I_n \\
\Rightarrow \quad \Omega_{2tr}\Omega_{2tr}' &= \lambda\Omega_{1tr}\Omega_{1tr}' \\
\Rightarrow \quad \Omega_2 &= \lambda\Omega_1,
\end{aligned}$$

which implies that the condition for identification fails. In other words, the two covariance matrices, once rescaled for the factor λ, contain the same amount of information. While λ can be used for estimating the variance of the structural shocks, the remaining part of information will be used for estimating the parameters of the conditional expected value of the structural form, i.e. A_0. However, this amount of information is the same as in standard bivariate SVARs, indicating that the results of Theorem 5 can be applied. This completes the proof. $\square$

C Frequentist and Bayesian estimators for reduced-form HVAR

C.1 GLS and ML estimators

The generalized least squares estimator can be easily obtained by using the compact notation for the model and for the covariance matrix of the error terms derived in Eq.s (39)(40). Using the well known formula for the GLS estimator we obtain

$$\begin{aligned}\hat{\phi}_{B,GLS} &= \left((X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n) \right)^{-1} \\ &\qquad (X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y. \end{aligned} \tag{75}$$

According to the partitioning of y and X as given in Eq. (41), the formula for the GLS estimator reported in Eq. (42) immediately follows.

The second part of this section, instead, is dedicated to transform the Gaussian likelihood function reported in Eq. (43), that we report here for practical reasons,

$$\begin{aligned} L(Y|\phi_B, \Omega_1, \Omega_2) &\propto |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \exp \Bigg\{ -\frac{1}{2} \big[y - (X' \otimes I_n)\phi_B \big]' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} \\ &\qquad \big[y - (X' \otimes I_n)\phi_B \big] \Bigg\}. \end{aligned} \tag{76}$$

in a more convenient way to derive the posterior distributions discussed in the next section. The term in the exponent can be written as

$$\begin{aligned} y' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y &- 2\phi_B'(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y + \\ &\phi_B'(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n)\phi_B. \end{aligned}$$

Using the mentionend partitioning of the X and y allows to write the addends as follows

$$\begin{aligned} y' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y &= y_1 (I_{T_1} \otimes \Omega_1)^{-1} y_1 + y_2 (I_{T_2} \otimes \Omega_2)^{-1} y_2 \\ \phi_B'(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} y &= \phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 + \phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 \\ \phi_B'(X' \otimes I_n)' \begin{pmatrix} I_{T_1} \otimes \Omega_1 & 0 \\ 0 & I_{T_2} \otimes \Omega_2 \end{pmatrix}^{-1} (X' \otimes I_n)\phi_B &= \phi_B'(X_1 X_1' \otimes \Omega_1^{-1})\phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1})\phi_B. \end{aligned}$$

Thus, an alternative expression for the likelihood function becomes

$$\begin{aligned} L(Y|\phi_B, \Omega_1, \Omega_2) \quad \propto \quad & |\Omega_1|^{-\frac{T_1}{2}} \, |\Omega_2|^{-\frac{T_2}{2}} \exp\Big\{ - \frac{1}{2} \big[y_1'(I_{T_1} \otimes \Omega_1)^{-1} y_1 + y_2'(I_{T_2} \otimes \Omega_2)^{-1} y_2 + \\ & -2\phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 - 2\phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 + \\ & + \phi_B'(X_1 X_1' \otimes \Omega_1^{-1})\phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1})\phi_B \big] \Big\}. \end{aligned}$$

C.2 Bayesian estimators

C.2.1 Case I) Inference on ϕ_B with Ω_1 and Ω_2 known

If Ω_1 and Ω_2 are known parameters, the kernel of the likelihood that is relevant for ϕ_B is

$$\begin{aligned} L(Y|\phi_B) \quad \propto \quad & \exp\Big\{ - \frac{1}{2} \big[- 2\phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 - 2\phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 + \\ & + \phi_B'(X_1 X_1' \otimes \Omega_1^{-1})\phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1})\phi_B \big] \Big\}. \end{aligned}$$

As a prior distribution for ϕ_B we can use

$$\phi_B \sim \mathcal{N}\big(\mu_\phi, \, V_\phi\big),$$

where $P(\phi_B) \propto \exp\big\{ - \frac{1}{2} \big[(\phi_B - \mu_\phi)' V_\phi^{-1} (\phi_B - \mu_\phi) \big] \big\}$, with the argument of the exponential function that can be written as

$$\big[(\phi_B - \mu_\phi)' V_\phi^{-1} (\phi_B - \mu_\phi) \big] = \phi_B' V_\phi^{-1} \phi_B - 2\phi_B' V_\phi^{-1} \mu_\phi + \mu_\phi' V_\phi^{-1} \mu_\phi,$$

where the last addend of the sum is not informative about ϕ_B.

The posterior distribution, thus, can be written as

$$\begin{aligned} P(\phi_B|Y) \quad \propto \quad & \exp\Big\{ - \frac{1}{2} \big[- 2\phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 - 2\phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 + \\ & + \phi_B'(X_1 X_1' \otimes \Omega_1^{-1})\phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1})\phi_B + \\ & + \phi_B' V_\phi^{-1} \phi_B - 2\phi_B' V_\phi^{-1} \mu_\phi \big] \Big\}. \end{aligned}$$

However, it is possible to show that:

$$\begin{aligned} \phi_B'(X_1 X_1' \otimes \Omega_1^{-1})\phi_B + \phi_B'(X_2 X_2' \otimes \Omega_2^{-1})\phi_B + \phi_B' V_\phi^{-1} \phi_B \quad &= \quad \phi_B' \underbrace{\big[(X_1 X_1' \otimes \Omega_1^{-1}) + (X_2 X_2' \otimes \Omega_2^{-1}) + V_\phi^{-1} \big]}_{V_\phi^{*-1}} \phi_B \\ &= \quad \phi_B' V_\phi^{*-1} \phi_B. \end{aligned}$$

Moreover:

$$\begin{aligned} & -2\phi_B'(X_1' \otimes \Omega_1^{-1})' y_1 - 2\phi_B'(X_2' \otimes \Omega_2^{-1})' y_2 - 2\phi_B' V_\phi^{-1} \mu_\phi \\ & \quad = -2\phi_B' \big[(X_1' \otimes \Omega_1^{-1})' y_1 + (X_2' \otimes \Omega_2^{-1})' y_2 + V_\phi^{-1} \mu_\phi \big] \\ & \quad = -2\phi_B' V_\phi^{*-1} \underbrace{V_\phi^* \big[(X_1' \otimes \Omega_1^{-1})' y_1 + (X_2' \otimes \Omega_2^{-1})' y_2 + V_\phi^{-1} \mu_\phi \big]}_{\mu_\phi^*} \\ & \quad = -2\phi_B' V_\phi^{*-1} \underbrace{\big[(X_1 X_1' \otimes \Omega_1^{-1}) + (X_2 X_2' \otimes \Omega_2^{-1}) + V_\phi^{-1} \big]^{-1} \big[(X_1' \otimes \Omega_1^{-1})' y_1 + (X_2' \otimes \Omega_2^{-1})' y_2 + V_\phi^{-1} \mu_\phi \big]}_{\mu^*} \\ & \quad = -2\phi_B' V_\phi^{*-1} \mu_\phi^*. \end{aligned}$$

If we add the term $\mu_\phi^{*\prime} V_\phi^{*-1} \mu_\phi^*$, that is however not informative for the parameter ϕ, the posterior is proportional to a Normal distribution

$$\phi_B|Y \sim \mathcal{N}\big(\mu_\phi^*,\, V_\phi^*\big)$$

where

$$\begin{aligned} V_\phi^* &= \Big[\big(X_1X_1' \otimes \Omega_1^{-1}\big) + \big(X_2X_2' \otimes \Omega_2^{-1}\big) + V_\phi^{-1}\Big]^{-1} \\ \mu_\phi^* &= V_\phi^*\big(V_\phi^{-1}\mu_\phi + (X_1' \otimes \Omega_1)'y_1 + (X_2' \otimes \Omega_2)'y_2\big). \end{aligned}$$

C.2.2 Case II) Inference on Ω_1 and Ω_2 with ϕ_B known

In the literature it is quite common to use inverse Wishart priors for covariance matrices. In our case we follow this approach and proceed in the same way for each of the two covariance matrices Ω_1 and Ω_2:

$$\begin{aligned} \Omega_1 &\sim i\mathcal{W}\big(S_1,\, d_1\big) \\ \Omega_2 &\sim i\mathcal{W}\big(S_2,\, d_2\big) \end{aligned} \tag{77}$$

with $P(\Omega_i) \propto |\Omega_i|^{-\frac{d_i+n+1}{2}} \exp\big\{-\frac{1}{2}\mathrm{tr}\,\big[\Omega_i^{-1}S_i\big]$, for $i = 1,\, 2$, and where $E(\Omega_i) = \frac{S_i}{d_i-n-1}$.

The first step consists in re-writing the likelihood function in a more convenient way. If the underlying model is written as

$$y = (X' \otimes I_n)\phi_B + u$$

the likelihood function is as in Eq. (43) before. The exponent can be re-written as

$$\begin{aligned} &\big[y_1(I_{T_1} \otimes \Omega_1^{-1})y_1 + \phi_B'(X_1X_1') \otimes \Omega_1^{-1})\phi_B - y_1(X_1' \otimes \Omega_1^{-1})\phi_B - \phi_B'(X_1 \otimes \Omega_1^{-1})y_1\big] \\ &\qquad \big[y_2(I_{T_2} \otimes \Omega_2^{-1})y_2 + \phi_B'(X_2X_2') \otimes \Omega_2^{-1})\phi_B - y_2(X_2' \otimes \Omega_2^{-1})\phi_B - \phi_B'(X_2 \otimes \Omega_2^{-1})y_2\big]. \end{aligned} \tag{78}$$

Using simple properties of the trace and vec operators, the first part under brackets can be written as

$$\begin{aligned} y_1(I_{T_1} \otimes \Omega_1^{-1})y_1 &= \mathrm{tr}\,\{Y'\Omega_1^{-1}Y_1\} = \mathrm{tr}\,\{\Omega_1^{-1}Y_1Y_1'\} \\ \phi_B'(X_1X_1' \otimes \Omega_1^{-1})\phi_B &= \mathrm{tr}\,\{B'\Omega_1^{-1}BX_1X_1'\} = \mathrm{tr}\,\{\Omega_1^{-1}BX_1X_1'B'\} \\ y_1(X_1' \otimes \Omega_1^{-1})y_1 &= \mathrm{tr}\,\{Y'\Omega_1^{-1}BX_1\} = \mathrm{tr}\,\{\Omega_1^{-1}BX_1Y_1'\} \\ \phi_B'(X_1\Omega_1^{-1})y_1 &= \mathrm{tr}\,\{B'\Omega_1^{-1}Y_1X_1'\} = \mathrm{tr}\,\{\Omega_1^{-1}Y_1X_1'B\} \end{aligned}$$

where we have used the decomposition

$$\underset{n\times T}{Y} = \big[\underset{n\times T_1}{Y_1} \quad \underset{n\times T_2}{Y_2}\big]$$

and the fact that $\phi_B = \mathrm{vec}\,(B)$. Obviously, these transformations can be replicated for the second part under the brackets of the exponent of the likelihood function. Definitely, the two exponents in Eq. (78) become

$$\begin{aligned} &\mathrm{tr}\,\big[\Omega_1^{-1}(Y_1Y_1' + BX_1X_1'B' - BX_1Y_1' - Y_1X_1'B)\big] + \mathrm{tr}\,\big[\Omega_2^{-1}(Y_2Y_2' + BX_2X_2'B' - BX_2Y_2' - Y_2X_2'B)\big] \\ &\quad = \mathrm{tr}\,\big[\Omega_1^{-1}(Y_1 - BX_1)(Y_1 - BX_1)'\big] + \mathrm{tr}\,\big[\Omega_2^{-1}(Y_2 - BX_2)(Y_2 - BX_2)'\big]. \end{aligned}$$

Now, if we combine the likelihood function with the two priors in Eq. (77) it becomes rather simple to derive the posterior distributions for the two variables Ω_1 and Ω_2. Overall, the joint posterior distribution for Ω_1 and Ω_2 can be written as

$$\begin{aligned} P(\Omega_1, \Omega_2|Y) &\propto P(\Omega_1)\, P(\Omega_1)\, P(Y|\Omega_1, \Omega_2) \\ &= |\Omega_1|^{-\frac{d_1+n+1}{2}} |\Omega_2|^{-\frac{d_2+n+1}{2}} |\Omega_1|^{-\frac{T_1}{2}} |\Omega_2|^{-\frac{T_2}{2}} \\ &\quad \exp\Big\{-\frac{1}{2}\mathrm{tr}\,(\Omega_1^{-1}S_1)\Big\}\exp\Big\{-\frac{1}{2}\mathrm{tr}\,(\Omega_2^{-1}S_2)\Big\} \\ &\quad \exp\Big\{-\frac{1}{2}\mathrm{tr}\,\big[\Omega_1^{-1}(Y_1 - BX_1)(Y_1 - BX_1)'\big]\Big\} \\ &\quad \exp\Big\{-\frac{1}{2}\mathrm{tr}\,\big[\Omega_2^{-1}(Y_2 - BX_2)(Y_2 - BX_2)'\big]\Big\}. \end{aligned} \tag{79}$$

However, focusing on the posterior distribution of each of the two covariance matrices, we obtain that

$$\begin{aligned} P(\Omega_1|Y) &\propto |\Omega_1|^{-\frac{T_1+d_1+n+1}{2}} \exp\Big\{-\frac{1}{2}\mathrm{tr}\,\big[\Omega_1^{-1}\big(S_1 + (Y_1 - BX_1)(Y_1 - BX_1)'\big)\big]\Big\} \\ P(\Omega_2|Y) &\propto |\Omega_2|^{-\frac{T_2+d_2+n+1}{2}} \exp\Big\{-\frac{1}{2}\mathrm{tr}\,\big[\Omega_2^{-1}\big(S_2 + (Y_2 - BX_2)(Y_2 - BX_2)'\big)\big]\Big\}, \end{aligned}$$

or, more compactly

$$\begin{aligned} \Omega_1 &\sim i\mathcal{W}\big(S_1^*,\, d_1^*\big) \\ \Omega_2 &\sim i\mathcal{W}\big(S_2^*,\, d_2^*\big), \end{aligned}$$

where

$$\begin{aligned} S_1^* &= S_1 + (Y_1 - BX_1)(Y_1 - BX_1)' = S_1 + \hat{\Omega}_{1,OLS}(T_1 - nm) \\ S_2^* &= S_2 + (Y_2 - BX_2)(Y_2 - BX_2)' = S_2 + \hat{\Omega}_{2,OLS}(T_2 - nm) \\ d_1^* &= T_1 + d_1 \\ d_2^* &= T_2 + d_2 \end{aligned}$$

and where

$$\hat{\Omega}_{i,OLS} = (Y_i - BX_i)(Y_i - BX_i)'/(T_i - nm), \quad \text{with} \quad i = \{1,\, 2\}. \tag{80}$$

D The test for identification via heteroskedasticity of Lütkepohl et al. (2020)

Lütkepohl et al. (2020) develop their test for identification via heteroskedasticity under the assumption that reduced-form error terms u_t have an elliptically symmetric distribution with density $\big(\sqrt{det\ \Omega_m}\big)^{-1} g(u_t'\Omega_m^{-1}u_t)$ where Ω_m is the covariance matrix in regime $m = 1, 2$, $g(.)$ is positive function such that the density integrates to one and the fourth moments of the distribution exist. A characteristic of elliptical distributions is that to impose the same kurtosis parameter for all the n elements of u_t. Formally, if we denote with ω_{im}^2 the i-th diagonal element of Ω_m, the kurtosis parameter $\kappa_m = \big[E(u_{it}^4)/3\omega_{im}^4\big] - 1$ is the same for all $i = 1, \ldots, n$ but can be different for different volatility regimes, m.

To implement the test of Lütkepohl et al. (2020), estimates of the kurtosis parameters are obtained as follows:

$$\hat{\kappa}_m = \frac{1}{3n}\sum_{i=1}^{n}\frac{z_i^m}{w_i^m} - 1, \quad m = 1, 2$$

with

$$z_i^m = \frac{\sum_{t\in T_m} (\hat{u}_{it} - \bar{u}_i^m)^4 - 6\hat{\omega}_i^4}{T_m - 4} \text{ and } w_i^m = \frac{T_m}{T_m - 1}\left(\hat{\omega}_i^4 - \frac{z_i^m}{T_m}\right) \quad m = 1, 2$$

where $\bar{u}_i = T_m^{-1} \sum_{t\in T_m} \hat{u}_{it}$ is the sample average of reduced-form residuals, $\hat{u}_{it}^m$, for the m-th regime, $T_1 = 1, \ldots, T_B$ and $T_2 = T_B + 1, \ldots, T$.

Denoting the estimated eigenvalues – ordered from largest to smallest – as $\hat{\lambda}_i$ for $i = 1, ..., n$ we write the test statistic as:

$$\begin{aligned} H_r(\hat{\kappa}_1, \hat{\kappa}_2) &= -c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2\, Tr \log\left(\frac{\prod_{k=s+1}^{s+r} \hat{\lambda}_k^{1/r}}{\frac{1}{r}\sum_{k=s+1}^{s+r} \hat{\lambda}_k}\right) \\ &= -c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2 \left[T \sum_{k=s+1}^{s+r} \log \hat{\lambda}_k - Tr \log\left(\frac{1}{r}\sum_{k=s+1}^{s+r} \hat{\lambda}_k\right)\right] \end{aligned}$$

with $s = 0, \ldots, n-1$ and $r = 2, \ldots, n-s$. Note that the first line of the equation highlights that the statistic is based on the ratio of the geometric mean to the arithmetic mean of the estimators of the eigenvalues assumed to be identical under the null. The term $c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2$ is defined as follows:

$$c(\tau, \hat{\kappa}_1, \hat{\kappa}_2)^2 = \left(\frac{1+\hat{\kappa}_1}{\tau} + \frac{1+\hat{\kappa}_2}{1-\tau}\right)^{-1} \quad \text{with } \tau \equiv T_B/T$$

Note that the fraction τ is assumed to be known and fixed. The test statistic converges in distribution to a $\chi^2((r+2)(r-1)/2)$ and involves the following pair of hypotheses:

$$H_0 : \lambda_{s+1} = \lambda_{s+2} = \ldots = \lambda_{s+r} \text{ against } H_1 : \neg H_0$$

where "¬" denotes negation.

Let us consider the case of testing identification via heteroskedasticity with $n = 3$ variables. Then we rely on $H_3(\hat{\kappa}_1, \hat{\kappa}_2)$ with a $\chi^2(5)$ distribution to test: $H_0 : \lambda_1 = \lambda_2 = \lambda_3$. If the null is rejected we test $H_0 : \lambda_1 = \lambda_2$ and $H_0 : \lambda_2 = \lambda_3$ using $H_2(\hat{\kappa}_1, \hat{\kappa}_2)$ with a $\chi^2(2)$ distribution. If also these hypotheses are rejected, the SVAR model is fully identified via heteroskedasticity.

E Empirical application – Further details and results

E.1 Data

The data entering the VAR model in Section VI are the following:

- $\Delta prod_t$ is percent change in world crude oil production and is defined as $100 \times \ln(prod_t/prod_{t-1})$. World oil production, $prod_t$, is sourced from the Monthly Energy Review maintained by the U.S. Energy Information Administration.

- The index of real economic activity, rea_t, is based on dry cargo ocean shipping rates and is available on the website of Lutz Kilian. It is used to proxy monthly changes in the world demand for industrial commodities, including crude oil.

- The real price of crude oil, rpo_t, is the refiner's acquisition cost of imported crude oil and it is available from the U.S. EIA. Deflation is carried out using the CPI for All Urban Consumers, as reported by the Bureau of Labor Statistics.

The time series included in the VAR and the reduced-form residuals of the VAR(6) are shown in Figures E.9 and E.10 that also displays a vertical bar in correspondence of the break date, October 1987.

Figure E.9: Data used in the SVAR model for the global market of crude oil (January 1973-December 2007)

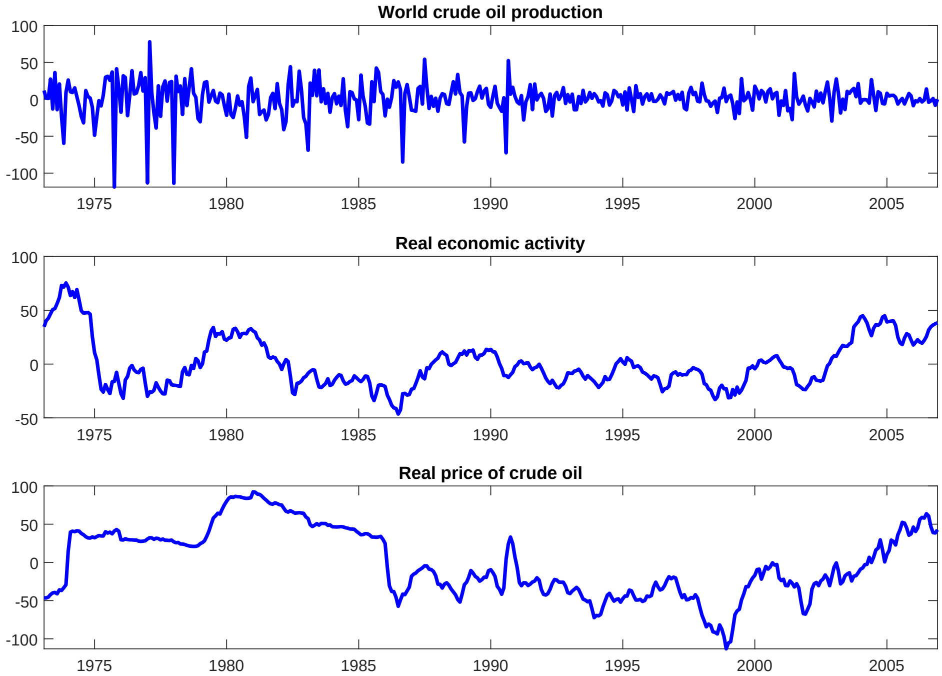
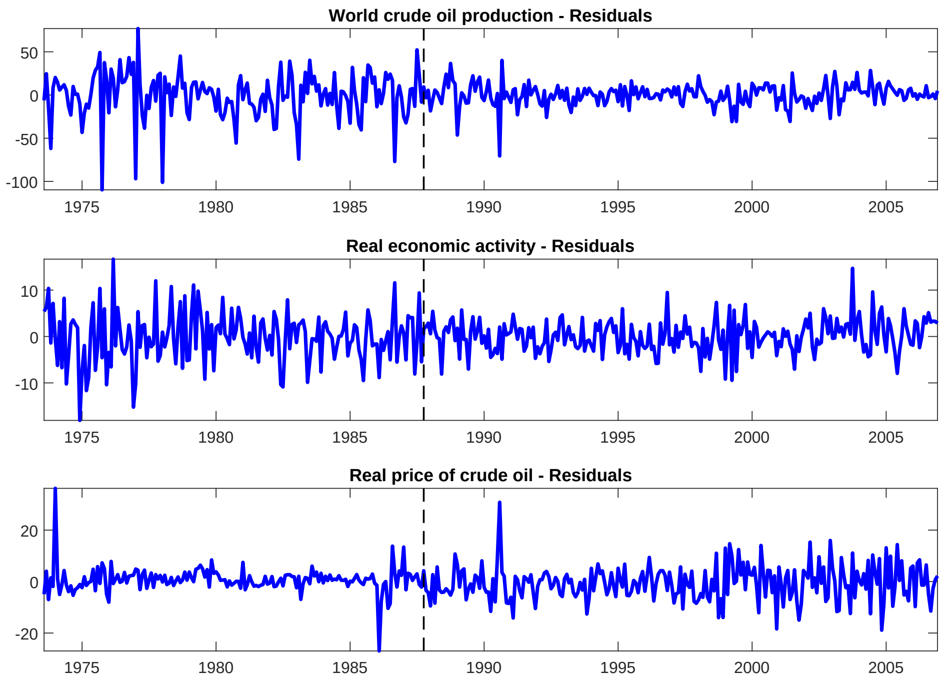
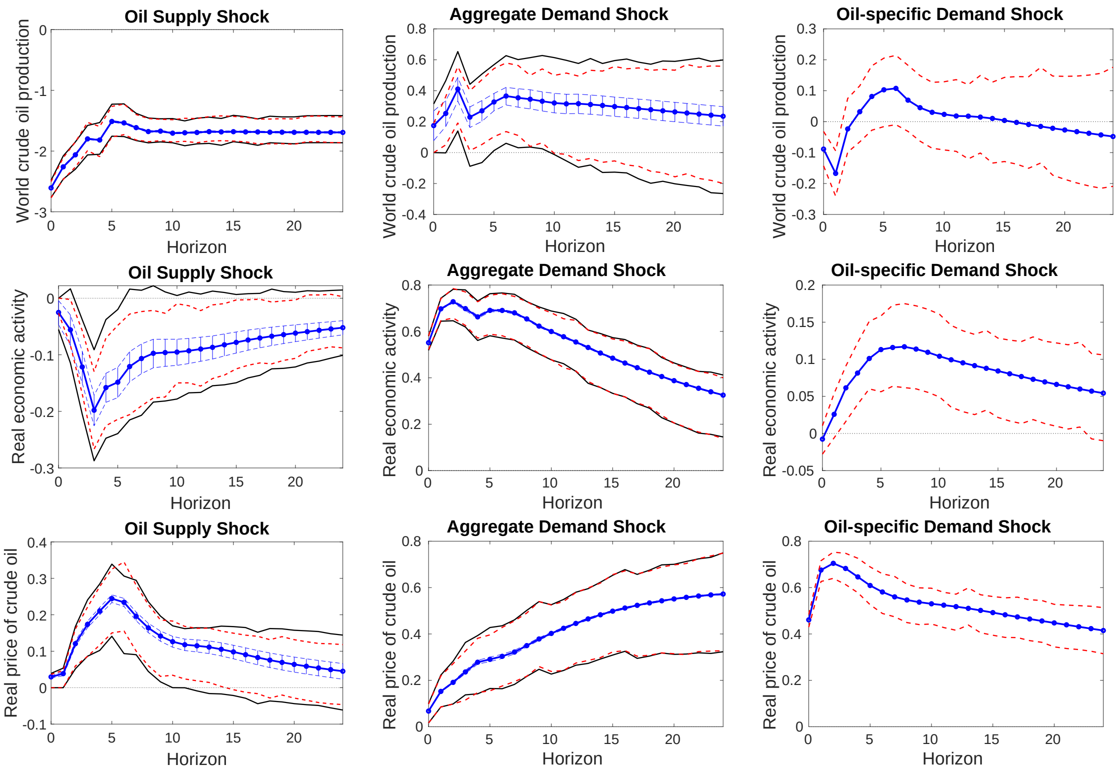


Figure E.10: Reduced form residuals and break date



Notes: Reduced form residuals and time of the break (1st October 1987). Monthly data.

Figure E.11: Impulse response functions $\mathcal{M}_2$ – Alternative implementation of Algorithm 1



Notes: the blue line with dots represents the standard Bayesian posterior mean response, the dashed red lines identify upper and lower bounds of the highest posterior density region with credibility 68%. Plots in first and second columns of the figure also report the set of posterior means (blue vertical bars) and the bounds of the robust credible region with credibility 68% (solid black curves). Identification via heteroskedasticity with multiple eigenvalues (i.e. only one shock is point identified), static and dynamic sign restrictions. We substitute Step 5 of Algorithm 1 with 10000 iterations of Step 4.1-Step 4.3. The interval $[\ell(\phi_m),\ u(\phi_m)]$ is then approximated by the minimum and maximum values over such iterations.